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To Thomas Stewart Wesner

Dad

To Margot

Phil

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CHAPTER 10

Systems of Linear Equations and Inequalities

In Sections 10–1, 10–2, and 10–3 we study what are called ''systems of n linear equations in n variables.'' These systems of equations are found in practically all areas where mathematics is used, including in electronics technology, engineering, economics, biology, business, etc. The study of these systems is extended and generalized in an area of mathematics called *linear algebra.*

Section 10–4 investigates systems of linear inequalities.[1] These appear widely also, but have their most common application in linear programming, a method of finding the ''best'' (i.e., cheapest, fastest, etc.) way to solve problems in business and management. Section 10–5 introduces the algebra of matrices, which finds application in science, engineering, business, and economics.

10–1 Solving systems of linear equations—the addition method

On Babylonian clay tablets, 4,000 years old, is found a problem that refers to quantities called a ''first silver ring'' and a ''second silver ring.'' If these quantities are represented by x and y, the tablet asks for the solution to the following system of equations: $\frac{x}{7} + \frac{y}{11} = 1$ and $\frac{6x}{7} = \frac{10y}{11}$. The answer is given as $\frac{x}{7} = \frac{11}{7 + 11} + \frac{1}{72}$ and $\frac{y}{11} = \frac{7}{7 + 11} - \frac{1}{72}$. Show that this answer is correct.

These two equations are called a system of two equations in two variables (unknowns). Obviously the solution of these systems has interested humankind for a long time. This is the topic of this section.

[1]Systems of nonlinear equations and inequalities are presented in section 11–4.

Systems in two variables

System of two linear equations in two variables

A system of two linear equations in two variables is a set of equations of the form

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where a_1 and b_1 are not both zero and a_2 and b_2 are not both zero. The values of a_1, a_2, b_1, b_2, c_1, and c_2 are constants.

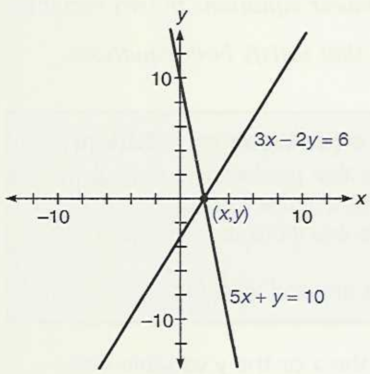


Figure 10–1

Systems of linear equations were described in something like this modern form at least 300 years ago. An example of such a system is

$$\begin{array}{ll} [1] & 3x - 2y = 6 \\ [2] & 5x + y = 10 \end{array}$$

The graph of each of these linear equations is a straight line; see figure 10–1.

In general, two straight lines intersect at a point, like (x,y), in the figure. In section 3–2 we saw that the *method of substitution for expression* can be used to eliminate one variable and find the point of intersection.

In this section we learn how to algebraically compute the value of the point (x,y) by the process called the addition method. The idea is to create a new equation in which one of the variables is not present using addition.

We could do this with these equations (see the following steps) by multiplying both members of equation [2] by the value 2, giving the new system equations [1] and [2a] (below). If we add the two left members of equations [1] and [2a] together, and the two right members together, we obtain equation [3].

$$\begin{array}{llcll} [1] & 3x - 2y = 6 & \quad & [1] & 3x - 2y = 6 \\ [2] & 5x + y = 10 & & [2a] & \underline{10x + 2y = 20} \\ & & & [3] & 13x \qquad = 26 \\ & & & & \qquad x = 2 \end{array}$$

We solved for x by eliminating y. We now use the same idea to obtain an equation in which x is not present; to do this we can multiply equation [1] by 5 and equation [2] by -3, giving the system [1b] and [2b], and equation [4].

$$\begin{array}{llcll} [1] & 3x - 2y = 6 & \quad & [1b] & 15x - 10y = 30 \\ [2] & 5x + y = 10 & & [2b] & \underline{-15x - 3y = -30} \\ & & & [4] & -13y = 0 \\ & & & & y = 0 \end{array}$$

The point (x,y) is $(2,0)$. We can verify this by checking that $(2,0)$ is a solution to both equations:

$$[1]\quad \begin{aligned} 3x - 2y &= 6 \\ 3(2) - 2(0) &= 6 \\ 6 &= 6\ \checkmark \end{aligned} \qquad\qquad [2]\quad \begin{aligned} 5x + y &= 10 \\ 5(2) - 0 &= 10 \\ 10 &= 10\ \checkmark \end{aligned}$$

An additional verification comes from the graph in figure 10–1, where the point of intersection appears to be at, or close to (2,0). In fact, modern graphing calculators make it easy to graph both straight lines and use the TRACE function to put the cursor over the solution, and thus obtain an approximation to the actual solution.

To generalize, *to solve a system of two linear equations* in two variables $\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$ means to *find all points (x,y) that satisfy both equations.*

> **To solve a system of two linear equations by addition**
> 1. Multiply each of the equations by values that produce two equivalent equations with the coefficients of x being additive inverses.
> 2. Add the left and right members of these equations to eliminate x.
> 3. Solve the resulting equation for y.
> 4. Repeat the process with the coefficients of y and solve for x.

Note One can choose to eliminate either the x or the y variable first.

Step 1 above requires multiplying each equation by certain values. These values are chosen to obtain the least common multiple (LCM) of the two coefficients. The **LCM** is the *smallest positive integer into which each of the (integer) coefficients divide.* Also, if the coefficients have the same sign we make one of the multipliers negative. This is summarized as follows.

> **Choosing a multiplier for each equation**
> 1. Choose multipliers that make the x (or y) coefficients the LCM of the coefficients.
> 2. If the coefficients have the same sign, make one multiplier negative.

This is illustrated in the following examples.

> **Notation $(n)\rightarrow$**
> In the examples, we use the notation $(n)\rightarrow$ to show that an equation was transformed by multiplying each term of both members by the integer n.

■ ***Example 10–1 A***

Solve the system by addition.

$$\begin{aligned} 2x - 3y &= -7 \\ 3x + 5y &= 1 \end{aligned}$$

Eliminate the x terms. The LCM of 2 and 3 is 6 (the smallest integer into which 2 and 3 divide). Since the values 2 and 3 have the same sign, choose one negative multiplier. Multiply the first equation by 3, and the second by -2. This is shown as follows.

$$\begin{array}{lll} 2x - 3y = -7 & (3)\rightarrow & 6x - 9y = -21 \\ 3x + 5y = 1 & (-2)\rightarrow & \underline{-6x - 10y = -2} \\ & & -19y = -23 \\ & & y = \dfrac{23}{19} \end{array}$$

Eliminate the y terms. The LCM of -3 and 5 is 15. The coefficients have opposite signs, so each multiplier can be positive.

$$\begin{array}{lll} 2x - 3y = -7 & (5)\rightarrow & 10x - 15y = -35 \\ 3x + 5y = 1 & (3)\rightarrow & \underline{9x + 15y = 3} \\ & & 19x \qquad = -32 \\ & & x = -\frac{32}{19} \end{array}$$

Thus, the solution is $(x,y) = (-\frac{32}{19}, \frac{23}{19})$ ■

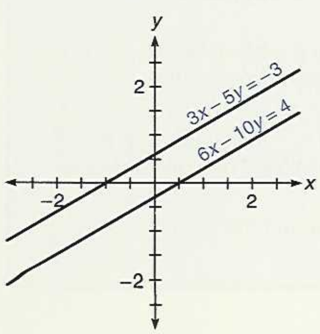
In example 10–1 A the two lines intersected at a point. However, two other things can happen. The lines may be parallel, and never meet, or the equations may represent the same line.

If the lines never meet (they are parallel) we say the system of equations is **inconsistent.** Algebraically we say that *a system of equations is inconsistent if there is no solution to that system.*

If the equations describe the same line, we say the system is **dependent.** Algebraically, *a system of equations is dependent if one equation can be derived from the rest of the equations.*

What happens in both cases is illustrated in example 10–1 B.

■ ***Example 10–1 B***



1. Solve the system $\begin{aligned} 6x - 10y &= 4 \\ 3x - 5y &= -3 \end{aligned}$.

To eliminate the x terms, we multiply the second equation by -2:

$$\begin{array}{r} 6x - 10y = 4 \\ \underline{-6x + 10y = 6} \\ 0 = 10 \end{array}$$

The figure shows why we obtained the false statement $0 = 10$—the two lines in the system are parallel, and therefore never meet. The system is inconsistent.

2. Solve the system $\begin{aligned} 2x - 3y &= 6 \\ -4x + 6y &= -12 \end{aligned}$.

Multiplying the first equation by 2:

$$\begin{aligned} 4x - 6y &= 12 \\ -4x + 6y &= -12 \\ \hline 0 &= 0 \end{aligned}$$

We obtain the statement $0 = 0$ because the first equation is just a multiple of the second equation (the first equation can be derived from the second equation). Both equations are the same line, and *the system is dependent.* The solution is therefore all points on the line $2x - 3y = 6$ (or $-4x + 6y = -12$).

As just illustrated, *an inconsistent system leads to a false statement. A dependent system leads to a statement that is true regardless of the value of x or y.*

Systems in more than two variables

A system of n equations involving n variables is a generalization of a system of two equations in two variables. Our objective is to find the values of all n variables that simultaneously satisfy all n equations. Our answer is written as an "n-tuple" (an ordered pair (x,y) is also called a "2-tuple"). It is possible for these systems to be dependent or inconsistent, just as with the case where $n = 2$ (as in example 10–1 B), although there will not always be a geometric interpretation of the result.

An example of a system of three equations involving three variables is

$$\begin{aligned} [1] \quad & x - 2y - z = 2 \\ [2] \quad & x + 4y + 2z = 2 \\ [3] \quad & -2x - 2y + z = -6 \end{aligned}$$

A system of n linear equations in n variables can be solved using the addition method. The idea is to *reduce the number of variables and equations, one step at a time.* A method that ensures finding a solution, when it exists, uses the idea of a "key equation."

To solve a system of *n* equations in *n* variables by addition

1. Select one equation as the key equation (see guideline below).
2. Use the key equation to eliminate one variable from all other equations.
3. Repeat steps 1 and 2 for this new system of $n - 1$ equations in $n - 1$ variables, until reaching 2 equations in 2 variables. Then go to step 4.
4. Obtain numerical values for two variables.
5. Substitute back into the key equation(s) to obtain the complete solution.

Note step 3 is only necessary for $n \geq 4$. A guideline for making the selection in steps 1 and 2 follows.

Selecting the key equation

Choose an equation in which the coefficient of one of the variables is 1. Use this key equation to eliminate the same variable from the other equations.

Of course there may be no equation in which any coefficient is one. In this case there may be no obvious choice as the key equation. This method is illustrated in example 10–1 C.

■ ***Example 10–1 C***

Solve the system of n equations in n variables.

1. [1] $x - 2y - z = 2$
[2] $x + 4y + 2z = 2$
[3] $-2x - 2y + z = -6$

Step 1: We select equation [1] as the key equation to eliminate x.

Step 2: Eliminate x from equation [2] using the key equation [1]:

$$\begin{array}{lll} [1] & x - 2y - z = 2 & \\ [2] & x + 4y + 2z = 2 & (-1)\rightarrow \end{array} \qquad \begin{array}{lr} & x - 2y - z = 2 \\ & \underline{-x - 4y - 2z = -2} \\ & -6y - 3z = 0 \\ [4] & 2y + z = 0 \end{array}$$

Divide each member by -3

Eliminate x from equation [3] using the key equation [1]:

$$\begin{array}{lll} [1] & x - 2y - z = 2 & (2)\rightarrow \\ [3] & -2x - 2y + z = -6 & \end{array} \qquad \begin{array}{lr} & 2x - 4y - 2z = 4 \\ & \underline{-2x - 2y + z = -6} \\ [5] & -6y - z = -2 \end{array}$$

Step 4: Equations [4] and [5] are a system of two equations in two variables, so solve as in example 10–1 A.

$$\begin{array}{ll} [4] & 2y + z = 0 \\ [5] & -6y - z = -2 \end{array}$$

$y = \frac{1}{2}, z = -1$

Step 5: Substitute for y and z in the key equation, [1].

$$\begin{array}{ll} [1] & x - 2y - z = 2 \\ & x - 2(\frac{1}{2}) - (-1) = 2 \\ & x = 2 \end{array}$$

Thus the solution is the ordered triple $(x,y,z) = (2,\frac{1}{2},-1)$.

2. Solve the system of equations for the solution (x,y,z):

[1] $2x - y + 3z = 0$
[2] $4x + 3y = 2$
[3] $2x + 2y - 3z = -3$

We can shorten the amount of work in this problem if we observe that the variable z is not present in equation [2]. Also the coefficients of z make elimination of z easy using equations [1] and [3]. We therefore choose equation [3] as the key equation to eliminate z from equation [1].

Step 1: Choose [3] as the key equation to eliminate z.

Step 2:
$$\begin{array}{ll} [1] & 2x - y + 3z = 0 \\ [3] & \underline{2x + 2y - 3z = -3} \\ [4] & 4x + y \phantom{{}-3z} = -3 \end{array}$$

Step 4: Equations [2] and [4] form a system of two equations and two variables.

[2] $4x + 3y = 2$
[4] $4x + y = -3$
$x = -\frac{11}{8}, y = \frac{5}{2}$

Step 5: Use the key equation [3] to find z:

[3]	$2x + 2y - 3z = -3$	
	$2(-\frac{11}{8}) + 2(\frac{5}{2}) - 3z = -3$	$x = -\frac{11}{8}, y = \frac{5}{2}$
	$-\frac{11}{4} + 5 - 3z = -3$	Perform arithmetic
	$-\frac{11}{4} + 8 = 3z$	Add +3 and $3z$ to each member
	$-11 + 32 = 12z$	Multiply each member by 4 to eliminate denominators
	$\frac{7}{4} = z$	

The solution is the 3-tuple $(x,y,z) = (-\frac{11}{8}, \frac{5}{2}, \frac{7}{4})$.

> We recognize that a system is dependent by obtaining the equation $0 = 0$ at some point; in this case, we will not attempt to describe the solution set, but simply state "dependent." If we arrive at a statement which is false, such as $0 = 2$, the system is inconsistent, and there are no points in the solution. We state "inconsistent" in this case.

3. Solve the system of equations.

[1] $x + 6y + 3z = 5$
[2] $x + 2y + z = 3$
[3] $4y + 2z = 2$

Use equation [2] as the key equation to eliminate x.

$$\begin{array}{llcl} [1] & x + 6y + 3z = 5 & & x + 6y + 3z = 5 \\ [2] & x + 2y + z = 3 \ (-1)\rightarrow & & \underline{-x - 2y - z = -3} \\ & & & \phantom{x+{}} 4y + 2z = 2 \\ & & [4] & \phantom{x+{}} 2y + z = 1 \quad \text{Divide by 2} \end{array}$$

Since equation [3] does not contain x, we do not modify it. We now use the system of two equations in two variables [3] and [4] to find y and z.

$$\begin{array}{lllr} [3] & 4y + 2z = 2 & & 4y + 2z = 2 \\ [4] & 2y + z = 1 & (-2)\rightarrow & -4y - 2z = -2 \\ \hline & & & 0 = 0 \end{array}$$

The statement $0 = 0$ tells us that *this system is dependent.* ■

Note It is possible for an inconsistent system to produce a statement like $0 = 0$ also. An example is the system $x + y + z = 1$, $2x + 2y + 2z = 2$, $x + y + z = 2$.

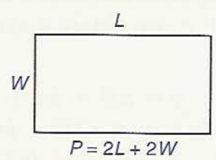
In this text we will not investigate systems of three or more variables that are dependent or inconsistent except to state the following. In an inconsistent system of n equations in n variables there are no solutions. In a dependent system of n equations in n variables there are infinitely many solutions. We will continue to focus our attention on systems that are independent and consistent. These systems have one solution.

There are many applied situations that lead to systems of linear equations.

■ Example 10–1 D

1. The golden ratio is a value that appears in many places throughout nature and art. Among other things, it has been observed that rectangles whose length and width are in this ratio are considered pleasing to the eye. This ratio is approximately 8 to 5.

   Now, suppose that an artist has a piece of stainless steel stock that is 14 feet long, and wishes to use the entire length to form a rectangle in the golden ratio. How long should the length and width be?

   For rectangles we know that perimeter $P = 2L + 2W$, L is length, and W is width (see the figure). For this case we know that $P = 14$, so we can write $14 = 2L + 2W$, or $7 = L + W$ (divide each member by 2). A ratio of 8 to 5 for length and width means

   $$\frac{8}{5} = \frac{L}{W}$$
   $$5L = 8W$$
   $$5L - 8W = 0$$

   Thus, we have a system of two equations in two variables: $\begin{aligned} L + W &= 7 \\ 5L - 8W &= 0 \end{aligned}$.
   Solving shows that the length should be $4\frac{4}{13}$ feet and the width should be $2\frac{9}{13}$ feet.

2. An investor invested a total of \$10,000 in a two-part mutual fund; one part is more risky than the other, but pays a higher return. The investor has forgotten how much was invested in each part of the fund, but knows that they paid 5% and 8% last year, and that the total income from the investment was \$695. Compute how much must have been invested at each rate.

Let x be the amount invested at 5%, and y the amount at 8%.

[1]	$x + y = 10{,}000$	The total of the two amounts is \$10,000
[2]	$0.05x + 0.08y = 695$	5% of x plus 8% of y totals \$695

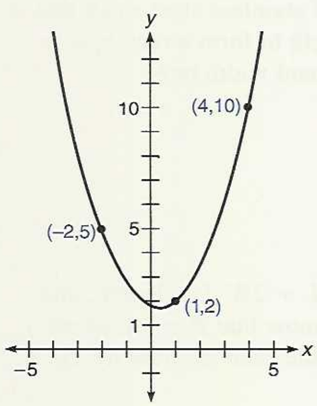
To eliminate decimals we multiply equation [2] by 100:

$$[2] \quad 5x + 8y = 69{,}500$$

Solving the system $\begin{array}{ll}[1] & x + y = 10{,}000 \\ [2] & 5x + 8y = 69{,}500\end{array}$ gives $x = 3{,}500$ and $y = 6{,}500$, so \$3,500 was invested at 5% and \$6,500 was invested at 8%.

3. A parabola is the graph of an equation of the form $y = ax^2 + bx + c$, $a \neq 0$ (section 4–1). Three noncollinear[2] points determine unique values of a, b, and c—that is, there is only one parabola that passes through any three noncollinear points. In computer modeling of geometric shapes, a computer might use a system of three equations in the three variables a, b, and c to find these unique values. A similar process occurs when a computer art program creates what are called Bezier curves.

Find the parabola that passes through the three points $(-2,5)$, $(1,2)$, and $(4,10)$.



Substituting the values for each point into the equation $y = ax^2 + bx + c$, we obtain a system of three equations in three variables.

$$\begin{array}{llll} & y = ax^2 + bx + c & & \\ x = -2, y = 5: & 5 = a(-2)^2 + b(-2) + c, & \text{or [1]} & 5 = 4a - 2b + c \\ x = 1, y = 2: & 2 = a(1)^2 + b(1) + c, & \text{or [2]} & 2 = a + b + c \\ x = 4, y = 10: & 10 = a(4)^2 + b(4) + c, & \text{or [3]} & 10 = 16a + 4b + c \end{array}$$

Solving produces $a = \frac{11}{18}$, $b = -\frac{7}{18}$, $c = \frac{16}{9}$, so the equation is $y = \frac{11}{18}x^2 - \frac{7}{18}x + \frac{16}{9}$. For illustration, the graph of this parabola is shown in the figure. ■

Mastery points

Can you

- Solve a system of n linear equations in n variables, giving the solution as an n-tuple or stating dependent or inconsistent as appropriate?
- Solve certain applications using systems of linear equations?

[2]Three points that are not on a straight line are said to be noncollinear.

Exercise 10–1

Solve the following systems of two equations in two variables; if the system is dependent or inconsistent state this.

1. $x + 10y = -7$, $-2x + 5y = 4$

2. $4x + 13y = 5$, $2x - 4y = -\frac{41}{13}$

3. $-3y = -6$, $2x + 7y = 4$

4. $x - 4y = -\frac{73}{6}$, $-6x - y = -2$

5. $\frac{2}{3}x + \frac{3}{5}x = \frac{14}{3}$, $x - \frac{2}{5}y = -6$

6. $-3x + 4y = -17$, $2x - \frac{3}{4}y = -4$

7. $-\frac{1}{2}x + 9y = -1$, $\frac{1}{2}x + \frac{3}{14}y = \frac{57}{14}$

8. $-6x + 12y = -15$, $-6x - 6y = 0$

9. $10x + 2y = 2$, $10y = -5$

10. $3x + 3y = 1$, $-6x - 6y = 2$

11. $6x + 12y = \frac{11}{2}$, $-3x - 6y = -\frac{11}{4}$

12. $-4x - \frac{2}{7}y = 10$, $-x - \frac{5}{7}y = -2$

13. $-2x - 3y = -10$, $x - 6y = 35$

14. $-2x - 8y = \frac{6}{7}$, $4x + 8y = -\frac{5}{7}$

15. $-\frac{2}{5}x + \frac{7}{9}y = 3$, $x + \frac{2}{3}y = 7$

16. $\frac{2}{7}x + 3y = -69$, $-x + \frac{2}{3}y = 7$

17. $4x + 4y = 12$, $9x + 9y = 18$

18. $-4x + 10y = 3$, $12x + 2y = 23$

19. $-2x - 6y = 6$, $4x + 12y = -12$

20. $4x - 6y = -\frac{27}{2}$, $7x + 18y = -\frac{33}{2}$

21. $-8x + 8y = -63$, $x + 4y = \frac{17}{2}$

22. $x + y = -2$, $-\frac{3}{2}x + \frac{13}{4}y = -28$

23. $-2x + 3y = -5$, $10x + 9y = 9$

24. $7x = \frac{1}{6}$, $6x - 5y = \frac{9}{14}$

Solve the following systems of three equations in three variables; if the system is dependent or inconsistent state this.

25. $x + y - 5z = -9$, $-x + y + 2z = 9$, $5x + 2y = -4$

26. $-5x - y + 3z = -14$, $-2x + 2y - 6z = 16$, $x + 7y + 2z = -5$

27. $-x - 6y - z = 22$, $\frac{1}{3}x + y - 3z = -9$, $2x - \frac{2}{5}y + z = 16$

28. $-5x + y + 9z = -37$, $4x - y - z = 11$, $\frac{9}{2}x + 2y - 5z = 24$

29. $-x + 3y - 3z = 29$, $x + \frac{4}{5}y - 5z = 30$, $-3x - 3y + \frac{12}{5}z = -30$

30. $-3x + \frac{8}{5}z = 14$, $5x - \frac{2}{3}y + 3z = 1$, $-x - 4y - 2z = -32$

31. $-3x + 6z = 6$, $\frac{9}{2}x + \frac{9}{2}y + 3z = 6$, $x + 7y - 2z = 40$

32. $-x + 6y + 6z = -11$, $-4x + \frac{2}{3}y + 4z = -4$, $5x + y + 2z = 10$

33. $y + z = 0$, $-3x + 2y + 2z = 0$, $6x + 2y - 6z = -40$

34. $-5x + 5y + 4z = 14$, $3x + y - 4z = -10$, $x - 2y + 2z = 0$

35. $x - y + z = -14$, $\frac{2}{5}x + 3y - 2z = 7$, $\frac{2}{5}x - 4y + 9z = -25$

36. $2x + z = 2$, $x + \frac{3}{10}y - 2z = 9$, $-5x + \frac{1}{5}y + 8z = -24$

Solve the following problems.

37. The perimeter of a certain rectangle is 44 centimeters long; if the ratio of length to width is 8 to 5, find the values of the length and width.

38. The ratio of length to width of a given rectangle is 3 to 2; if the perimeter is 100 inches, find each dimension.

39. The length of a rectangle is 5 inches longer than the width. The perimeter is 36 inches. Find the two dimensions.

40. The length of a certain rectangle is 3 inches longer than twice the width. The perimeter is 120 inches. Find the two dimensions.

41. The width of a rectangle is 10 millimeters less than half the length. If the perimeter is 150 millimeters, find the length and width.

42. The ratio of width to length of a certain rectangle is 1.5 to 4.0. If the perimeter of the rectangle is 25 centimeters, find the length and width.

43. An investor invested a total of \$12,000 into a two-part mutual fund; one part paid 5% and the other part paid 10% last year. If the total income from the investments was \$800, compute how much must have been invested at each rate.

44. A total of \$20,000 was invested, part at 4% and the rest at 12%. The total income from both investments was \$2,000. How much was invested at each rate?

45. \$900 was earned on \$10,000 last year; part of the \$10,000 was invested at 6% and the rest at 12%. How much was invested at each rate?

46. If the total income from a \$40,000 investment last year was \$5,000, and the money was invested in two funds paying 6% and 16%, how much was invested at each rate?

47. A parabola is the graph of an equation of the form $y = ax^2 + bx + c$. Find the values of a, b, and c so that the parabola will pass through the points $(-4,1)$, $(0,3)$, and $(2,6)$.

48. Find the equation of the parabola that passes through the points $(-1,1)$, $(\frac{1}{2},3)$ and $(2,10)$. See problem 47.

49. A nonvertical straight line is the graph of an equation of the form $y = mx + b$. Two points uniquely determine a straight line. Find the equation of the straight line that passes through the points $(5,-1)$ and $(8,6)$.

50. Find the equation of the straight line that passes through the points $(-2,8)$ and $(5,1)$. See problem 49.

51. A straight line passes through the points $(-2,3)$ and $(1,5)$; a second straight line passes through the points $(0,2)$ and $(4,1)$. Find the point at which these two straight lines intersect. See problem 49.

52. Find the equation of the straight line that passes through the points $(2,5)$ and $(6,8)$. See problem 49.

53. Find the equation of the straight line that passes through the points $(-2,-3)$ and $(0,2)$. See problem 49.

54. On Babylonian clay tablets, 4,000 years old, is found a problem that refers to quantities called a "first silver ring" and a "second silver ring." If these quantities are represented by x and y, the tablet asks for the solution to the following system of equations: $\frac{x}{7} + \frac{y}{11} = 1$ and $\frac{6x}{7} = \frac{10y}{11}$. The answer is given as $\frac{x}{7} = \frac{11}{7+11} + \frac{1}{72}$ and $\frac{y}{11} = \frac{7}{7+11} - \frac{1}{72}$. Show that this answer is correct.

Skill and review

1. Simplify $\sqrt{\frac{4x^2}{27y^3z}}$.

2. Solve $\frac{2x-1}{3} = \frac{5-3x}{4}$.

3. Solve $\frac{2x-1}{3} = \frac{5-3x}{x}$.

4. Solve $\left|\frac{2x-1}{3}\right| > 5$.

5. Solve $\left|\frac{2x-1}{x}\right| < 5$.

10–2 Systems of linear equations—matrix elimination

A certain mix of animal feed contains 10% protein; a second mix is 45% protein. How many pounds of each must be mixed to obtain 250 pounds of a 30% protein mix?

This section discusses another method of solving systems of linear equations. The stated problem could be solved by such a system.

The choice of symbols used for the variables in a system of equations is not important; it is the coefficients that determine the solution. A matrix is a mathematical tool that allows us to focus exclusively and efficiently on these coefficients.

Matrix
A matrix is a rectangular array of numbers. The numbers that make up the matrix are called its elements.

The words matrix and array are used to mean the same thing. Matrices are shown by enclosing the array in brackets, as illustrated by the following examples.

$$\overset{\text{I}}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \qquad \overset{\text{II}}{\begin{bmatrix} 4 & -7 & 3 \\ 2 & 3 & -4 \\ 3 & 1 & 0 \end{bmatrix}} \qquad \overset{\text{III}}{\begin{bmatrix} \sqrt{2} & -1 & 0 \\ 8 & 0 & 5 \end{bmatrix}} \qquad \overset{\text{IV}}{\begin{bmatrix} 9 \\ -2 \\ \pi \\ 5 \end{bmatrix}}$$

Matrices are classified by stating the numbers of rows and columns they contain, *always stating the number of rows first.* The examples above are:

I 2×2	("two by two")	(two rows by two columns)
II 3×3	("three by three")	(three rows by three columns)
III 2×3	("two by three")	(two rows by three columns)
IV 4×1	("four by one")	(four rows by one column)

When the number of rows equals the number of columns the matrix is said to be **square.** Examples I and II are square matrices; square matrices are also described as being of an **order.** Example I is an order 2 matrix, and example II is an order 3 matrix.

A symbolic definition[3] of a matrix A with m rows and n columns is

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}$$

We usually use capital letters to denote a matrix, and the corresponding lower case letter, with subscripts, to denote the elements of that array. Thus an element of A is $a_{i,j}$, $1 \le i \le m$, $1 \le j \le n$. We read this element as "the i-j'th element of A" or "a sub ij." For example $a_{2,3}$ is "a sub two three," the element in row 2 and column 3 of matrix a.

Historically, matrices developed as a shorthand notation useful in describing and solving systems of equations. In solving systems of linear equations by the addition method (section 10–1) we focus our attention on the coefficients of the variables, not the letter that represents the variable. In other words, we would solve both of the following systems in the same way.

$$\begin{aligned} 2x - 3y &= 5 \\ 3x + y &= 13 \end{aligned} \quad \text{or} \quad \begin{aligned} 2a - 3b &= 5 \\ 3a + b &= 13 \end{aligned}$$

The information in both systems is contained in the 2×3 matrix

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & 13 \end{bmatrix}$$

Each row of the matrix corresponds to one of the equations. The first two columns of the matrix correspond to the coefficients of the variables x and y, and the third column corresponds to the constants.

[3]This modern notation was perfected by C. E. Cullis in his book *Matrices and Determinoids,* (Cambridge), Vol. I (1913).

The method of matrix elimination parallels the steps we could do to solve this system of equations by addition. To illustrate matrix elimination we will solve this system, using the matrix shown.

First we could multiply the second row by 3 and add it to the first row (this will eliminate the value -3). This is similar to using a key equation in section 10–1. This would produce the matrix

$$\begin{bmatrix} 3(3)+2 & 3(1)+(-3) & (13)+5 \\ 3 & 1 & 13 \end{bmatrix} \text{ which is } \begin{bmatrix} 11 & 0 & 44 \\ 3 & 1 & 13 \end{bmatrix}$$

We could divide the first row by 11. This is the same as saying that $11x + 0y = 44$, so $x + 0y = 4$.

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 13 \end{bmatrix}$$

If we now multiply the first row by -3 and add the result to the second row we obtain

$$\begin{bmatrix} 1 & 0 & 4 \\ (-3+3) & 0+1 & (-12+13) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix now corresponds to the system of equations $\begin{aligned} x + 0y &= 4 \\ 0x + y &= 1, \end{aligned}$ or $\begin{aligned} x &= 4 \\ y &= 1 \end{aligned}$, which has the point (4,1) for its solution. This solution is the rightmost column of the matrix. This is also the solution to the original system of equations.

Before we further discuss solving systems of linear equations by this method we need some more definitions. The order 2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in the variable columns of $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$ above is called the order 2 **identity matrix.**

The order 3 identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and in general the **order *n* identity matrix** has ones on the **main diagonal** (upper-left to lower-right) and zeros everywhere else.

An **augmented matrix** is a matrix formed as we did above that represents a system of n equations in n variables. It has n rows (one for each equation) and $n + 1$ columns—the first n columns contain the coefficients of the variables in the equations, and the last column contains the constants. This matrix can be used to obtain the solutions to the system of equations.

The elimination method uses the following operations, called **row operations.** It can be proven that *row operations do not change the solution set of a system of equations.*

Row operations

1. Multiply or divide each entry of a row by a nonzero value.
2. Add a nonzero multiple of one row to a nonzero multiple of another row, and replace either row by the result.
3. Rearrange the order of the rows.

The **matrix elimination method** for solving a system of n equations in n variables consists of the following steps.

1. Create the augmented matrix.
2. Use the row operations to obtain the identity matrix in the first n columns.
3. The solution is in the column of constants.

To obtain the identity matrix we will **sweep out** each of the first n columns. This means to use one nonzero value in a column to eliminate the rest of the nonzero elements in that column.

We will use the idea of a *key row* (KR) for sweeping out columns; each row will serve as a key row once (unless we simply rearrange rows). Using key rows helps keep track of what we have done; without this idea the procedure can become confusing. We will see that this is the same concept as "key equations" from section 10–1. Remember that *a row may be a key row only once.*

It is helpful to box in the element in the key row that is also in the column we are sweeping out. This is shown in example 10–2 A.

■ ***Example 10–2 A***

Solve each system of equations.

1. $3x - 2y = 8$
$-2x - 5y = 1$

Step 1: The augmented matrix is $\begin{bmatrix} 3 & -2 & 8 \\ -2 & -5 & 1 \end{bmatrix}$.

Step 2: We now use row operations to sweep out column 1. We let the first row be the key row, so we box in the element in row 1, column 1. Our objective is to make the entry in row 2, column 1, zero.

As shown below it is a good idea to use a "scratch pad" to perform the arithmetic.

$$\begin{bmatrix} \boxed{3} & -2 & 8 \\ -2 & -5 & 1 \end{bmatrix}$$

Add twice the first row to three times the second

2R1:	6	−4	16
3R2:	−6	−15	3
	0	−19	19 → Row 2

$$\begin{bmatrix} 3 & -2 & 8 \\ 0 & -19 & 19 \end{bmatrix}$$

The first column is swept out

$$\begin{bmatrix} 3 & -2 & 8 \\ 0 & 1 & -1 \end{bmatrix}$$ Divide the second row by -19

$$\begin{bmatrix} 3 & -2 & 8 \\ 0 & \boxed{1} & -1 \end{bmatrix}$$ Now the second row is the key row, and we want to sweep out the second column; box in the element in row 2 column 2; add twice the second row to the first row

$$\begin{array}{rrrrl} \text{R1:} & 3 & -2 & 8 & \\ \text{2R2:} & 0 & 2 & -2 & \\ & 3 & 0 & 6 & \to \text{Row 1} \end{array}$$

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}$$ The second column is now swept out

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$ Divide the first row by 3

The third column gives the solution $(x,y) = (2,-1)$.

2. $2x - y + z = 10$
$x + 2y - z = -3$
$3x + y + 2x = 11$

The augmented matrix is $\begin{bmatrix} 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & -3 \\ 3 & 1 & 2 & 11 \end{bmatrix}$.

Let us first focus on column 3, the "z" column. Since column 3 has a 1 in row 1, we choose this row to be our key row. We will sweep out column 3 using row 1, so we box in the element in row 1 column 3. We will add multiples of row 1 to multiples of the other rows to make the entries in column 3 in those rows zero.

We can make the entry in row 2 zero by adding row 1 (the key row) to row 2; "scratch pad" work is shown below the matrices.

$$\begin{bmatrix} 2 & -1 & \boxed{1} & 10 \\ 1 & 2 & -1 & -3 \\ 3 & 1 & 2 & 11 \end{bmatrix} \begin{array}{l} \text{Key row} \\ \\ \\ \end{array} \quad \text{becomes} \quad \begin{bmatrix} 2 & -1 & \boxed{1} & 10 \\ 3 & 1 & 0 & 7 \\ 3 & 1 & 2 & 11 \end{bmatrix}.$$

$$\begin{array}{rrrrrl} \text{R1:} & 2 & -1 & 1 & 10 & \\ \text{R2:} & 1 & 2 & -1 & -3 & \\ & 3 & 1 & 0 & 7 & \to \text{New row 2} \end{array}$$

Let us indicate the last step by the notation R2 $\leftarrow$ KR + R2, which shows that "row 2 (R2) is replaced by the sum of the key row (KR) and row 2 (R2)."

We now make the last entry in column 3 zero by adding -2 times row 1 to row 3, and putting the result in row 3:

$$\begin{bmatrix} 2 & -1 & \boxed{1} & 10 \\ 3 & 1 & 0 & 7 \\ 3 & 1 & 2 & 11 \end{bmatrix} \begin{array}{l} \text{Key row} \\ \\ \\ \end{array} \quad \text{becomes} \quad \begin{bmatrix} 2 & -1 & \boxed{1} & 10 \\ 3 & 1 & 0 & 7 \\ -1 & 3 & 0 & -9 \end{bmatrix}.$$

$$\begin{array}{rrrrrl} -2\text{R1:} & -4 & 2 & -2 & -20 & \\ \text{R3:} & 3 & 1 & 2 & 11 & \\ \hline & -1 & 3 & 0 & -9 & \to \text{R3} \end{array}$$

Our notation for this is R3 $\leftarrow -2(\text{KR}) + \text{R3}$. *It is important to note that column 3 is swept out.* This means there is only one nonzero entry in that column. *Also, row 1 may not be reused as a key row.* (Assuming we do not change the order of the rows.)

Now sweep out column 2; we want to arrange column 2 so that all entries but one are zero. We first choose the next key row. Row 2 would be a good choice because column 2 has an entry of 1 in row 2, which will make our arithmetic that much easier. Thus, we box in the entry in column 2 and row 2.

$$\begin{bmatrix} 2 & -1 & 1 & 10 \\ 3 & \boxed{1} & 0 & 7 \\ -1 & 3 & 0 & -9 \end{bmatrix} \text{ becomes } \begin{bmatrix} 5 & 0 & 1 & 17 \\ 3 & \boxed{1} & 0 & 7 \\ -10 & 0 & 0 & -30 \end{bmatrix}.$$

The "scratch pad" work is:

$$\begin{array}{lrrrrl} \text{R1:} & 2 & -1 & 1 & 10 & \\ \text{KR:} & 3 & 1 & 0 & 7 & \\ \hline & 5 & 0 & 1 & 17 & \rightarrow \text{R1} \end{array} \qquad \begin{array}{lrrrrl} -3\text{KR:} & -9 & -3 & 0 & -21 & \\ \text{R3:} & -1 & 3 & 0 & -9 & \\ \hline & -10 & 0 & 0 & -30 & \rightarrow \text{R3} \end{array}$$

Observe that we can divide each entry in row 3 by -10 to obtain

$$\begin{bmatrix} 5 & 0 & 1 & 17 \\ 3 & 1 & 0 & 7 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Now we sweep out column 1; only row 3 is left to serve as a key row.

$$\begin{bmatrix} 5 & 0 & 1 & 17 \\ 3 & 1 & 0 & 7 \\ \boxed{1} & 0 & 0 & 3 \end{bmatrix} \begin{array}{l} -5(\text{KR}) + \text{R1} \rightarrow \\ -3(\text{KR}) + \text{R2} \rightarrow \\ \text{Key row} \end{array} \quad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Note The notation above states that we added -5 times the key row to row 1, and -3 times the key row to row 2, and the key row is row 3.

If we now rearrange the order of the rows we obtain the 3×3 identity matrix and the values of x, y, and z in the last column:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

This matrix corresponds to the set of equations $x = 3$, $y = -2$, and $z = 2$. Thus, our solution is $(x,y,z) = (3,-2,2)$.

3. $2x - y + w = 10$
$x + 2y - z = 3$
$y - z + 3w = -2$
$3x + 4w = 5$
Put in zero coefficients where a variable does not appear in an equation. This gives the augmented matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 10 \\ 1 & 2 & -1 & 0 & 3 \\ 0 & 1 & -1 & 3 & -2 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix}$$

Since there are only two nonzero entries in column 3 we will sweep it out first.

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 10 \\ 1 & 2 & \boxed{-1} & 0 & 3 \\ 0 & 1 & -1 & 3 & -2 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix} \begin{array}{l} \\ \text{Key row} \\ -\text{KR} + \text{R3} \rightarrow \\ \\ \end{array} \begin{bmatrix} 2 & -1 & 0 & 1 & 10 \\ 1 & 2 & -1 & 0 & 3 \\ -1 & -1 & 0 & 3 & -5 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix}$$

Columns 2 or 4 are good choices to be swept out next. Column 1 is the poorest choice because it has the fewest zeros. We choose column 4 next, using row 1 as the key row.

$$\begin{bmatrix} 2 & -1 & 0 & \boxed{1} & 10 \\ 1 & 2 & -1 & 0 & 3 \\ -1 & -1 & 0 & 3 & -5 \\ 3 & 0 & 0 & 4 & 5 \end{bmatrix} \begin{array}{l} \text{Key row} \\ \\ -3(\text{KR}) + \text{R3} \rightarrow \\ -4(\text{KR}) + \text{R4} \rightarrow \end{array} \begin{bmatrix} 2 & -1 & 0 & 1 & 10 \\ 1 & 2 & -1 & 0 & 3 \\ -7 & 2 & 0 & 0 & -35 \\ -5 & 4 & 0 & 0 & -35 \end{bmatrix}$$

Only rows 3 and 4 may now be used as key rows.

We will sweep out column 2, using row 3 as the key row.

$$\begin{bmatrix} 2 & -1 & 0 & 1 & 10 \\ 1 & 2 & -1 & 0 & 3 \\ -7 & \boxed{2} & 0 & 0 & -35 \\ -5 & 4 & 0 & 0 & -35 \end{bmatrix} \begin{array}{l} \text{KR} + 2(\text{R1}) \rightarrow \\ -\text{KR} + \text{R2} \rightarrow \\ \text{Key row} \\ -2(\text{KR}) + \text{R4} \rightarrow \end{array} \begin{bmatrix} -3 & 0 & 0 & 2 & -15 \\ 8 & 0 & -1 & 0 & 38 \\ -7 & 2 & 0 & 0 & -35 \\ 9 & 0 & 0 & 0 & 35 \end{bmatrix}$$

We can now sweep out column 1; only row 4 is left to serve as a key row.

$$\begin{bmatrix} -3 & 0 & 0 & 2 & -15 \\ 8 & 0 & -1 & 0 & 38 \\ -7 & 2 & 0 & 0 & -35 \\ \boxed{9} & 0 & 0 & 0 & 35 \end{bmatrix} \begin{array}{l} \text{KR} + 3(\text{R1}) \rightarrow \\ 8(\text{KR}) - 9(\text{R2}) \rightarrow \\ 7(\text{KR}) + 9(\text{R3}) \rightarrow \\ \text{Key row} \end{array} \begin{bmatrix} 0 & 0 & 0 & 6 & -10 \\ 0 & 0 & 9 & 0 & -62 \\ 0 & 18 & 0 & 0 & -70 \\ 9 & 0 & 0 & 0 & 35 \end{bmatrix}$$

At this point all of the columns have been swept out. Each has all but one zero entries, and none of these nonzero entries is in the same row.

Rearrange the order of the rows so the nonzero entries are on the main diagonal.

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 35 \\ 0 & 18 & 0 & 0 & -70 \\ 0 & 0 & 9 & 0 & -62 \\ 0 & 0 & 0 & 6 & -10 \end{bmatrix}$$

Divide each row by its nonzero entry in columns 1 through 4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{35}{9} \\ 0 & 1 & 0 & 0 & -\frac{35}{9} \\ 0 & 0 & 1 & 0 & -\frac{62}{9} \\ 0 & 0 & 0 & 1 & -\frac{5}{3} \end{bmatrix}$$

The solution is the 4-tuple $(w,x,y,z) = (\frac{35}{9}, -\frac{35}{9}, -\frac{62}{9}, -\frac{5}{3})$. ■

The example 10–2 B illustrates what happens when a system is dependent or inconsistent.

■ ***Example 10–2 B*** Solve each system of equations using matrix elimination.

1. $x + y - z = -2$
$2x - y + 2z = 6$
$3x + z = 3$

Augmented matrix
$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 2 & -1 & 2 & 6 \\ 3 & 0 & 1 & 3 \end{bmatrix}$$

Swept out matrix
$$\begin{bmatrix} 3 & 0 & 1 & 4 \\ 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row of the swept out matrix corresponds to the statement $0x + 0y + 0z = 1$, or $0 = 1$. This is a false statement, which means that there is no solution to this system. This system of equations is *inconsistent.* Thus there is no solution.

Note Whenever a system is inconsistent, at least one row will contain a false statement such as $0 = 1$ in the last example.

2. $2x - 3y + z = 4$
$x + y - 2z = -3$
$3x - 2y - z = 1$

Augmented matrix
$$\begin{bmatrix} 2 & -3 & 1 & 4 \\ 1 & 1 & -2 & -3 \\ 3 & -2 & -1 & 1 \end{bmatrix}$$

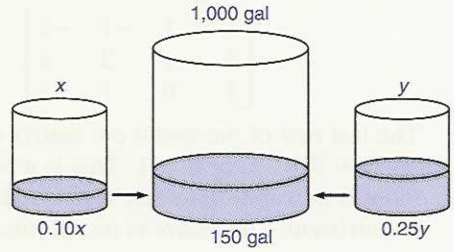
Swept out matrix
$$\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If there are one or more rows of zeros, when we have swept out as many columns as nonzero rows, and there is no false statement (such as $1 = 0$), the system is dependent but consistent (not inconsistent). In this case we have swept out two columns and there are two nonzero rows. Thus, this system is dependent.

Note There are other methods of determining when a system is dependent or inconsistent than those shown here. Section 10–3 shows a simpler method, for example. Thus we will not dwell on this detail here. ■

Matrix elimination provides another way to solve systems of linear equations when they occur in applications.

Example 10–2 C A chemical company stores a certain herbicide in two concentrations of herbicide and water: 10% solution and 25% solution. It needs to manufacture 1,000 gallons of a 15% solution by mixing the correct amounts of the 10% and 25% solutions. How many gallons of each of the 10% and 25% solutions should be mixed to obtain the required product?



This type of problem can be solved by focusing on two aspects of the problem: the total amounts of the solutions, and the total amounts of the herbicide itself. For example, in the 10% solution we know that 10% of the total is herbicide and the rest water; thus, if x is the amount of the 10% solution then 10% of x ($0.10x$) is herbicide.

Let x be the amount of the 10% solution, and y be the amount of the 25% solution. Then $x + y = 1{,}000$, since the total amount will be 1,000 gallons.

Now, focus on the herbicide itself. We need 1,000 gallons of a 15% solution; this will contain 15% of 1,000, or 150, gallons of herbicide. This 150 gallons of herbicide comes from 10% of x and 25% of y.

$$\begin{aligned} 0.10x + 0.25y &= 150 \\ 10x + 25y &= 15{,}000 \\ 2x + 5y &= 3{,}000 \end{aligned}$$

We now have two equations in two variables.

$$\begin{aligned} [1] \quad x + y &= 1000 \\ [2] \quad 2x + 5y &= 3{,}000 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 1{,}000 \\ 2 & 5 & 3{,}000 \end{array}\right]$$

Solution matrix

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{2{,}000}{3} \\ 0 & 1 & \frac{1{,}000}{3} \end{array}\right]$$

The solution is $x = \frac{2{,}000}{3}$, $y = \frac{1{,}000}{3}$. Thus the final product should consist of $666\frac{2}{3}$ gallons of the 10% solution and $333\frac{1}{3}$ gallons of the 25% solution. ■

Mastery points

Can you

- Solve a system of n linear equations in n variables by matrix elimination?
- Describe a dependent solution set?
- Solve applications whose solutions can be found by n linear equations in n variables?

Exercise 10–2

Solve the following systems of equations by matrix elimination; after describing the solution set, state dependent or inconsistent if appropriate.

1. $2x + \frac{2}{3}y = -2$
$-3x + y = 15$

2. $2x + \frac{1}{2}y = 9$
$-3x + y = -10$

3. $-2x + 5y = 9$
$x + 3y = \frac{13}{2}$

4. $2x + 3y = 7$
$-5x + 2y = 11$

5. $-3x + 4y = -1$
$4x + y = 14$

6. $-3x + 2y = -2$
$3x + 2y = 0$

7. $\frac{2}{5}x + \frac{1}{3}y = 1$
$-2x + 2y = -16$

8. $\frac{1}{3}x + \frac{1}{4}y = 2$
$-2x - y = -16$

9. $\frac{3}{4}x + 2y = 0$
$-3x + y = -27$

10. $2x + y = 0$
$3x - 5y = -65$

11. $\frac{1}{2}x + 3y = 21$
$2x - 2y = 0$

12. $-3x + 2y = 3$
$6x + y = 9$

13. $4x + 5y = -17$
$-x + 10y = 38$

14. $4x - 2y = 8$
$\frac{1}{2}x + 3y = 27$

15. $-2x + \frac{1}{3}y = 14$
$4x + \frac{2}{3}y = -20$

16. $6x - 2y = -9$
$8x + 5y = 11$

17. $3x + y = 20$
$-2x + 2y = 0$

18. $2x + 3y = 8$
$-4x - 6y = -16$

19. $-3x + y = -4$
$6x + 2y = 20$

20. $2x + 5y = 1$
$5x - y = -11$

21. $\frac{2}{3}x - 3y = 10$
$-2x + y = -14$

22. $4x - 3y = 0$
$-8x + 3y = -5$

23. $x + y - 5z = -9$
$-x + y + 2z = 9$
$5x + 2y = -4$

24. $-5x - y + 3z = -14$
$-2x + 2y - 6z = 16$
$x + 7y + 2z = -5$

25. $-x - 6y - z = 22$
$x + y - 3z = -5$
$2x - 2y + z = 24$

26. $-5x + y + 2z = -16$
$4x - y - z = 11$
$x + 2y - 5z = 17$

27. $-x + 3y - 3z = 15$
$x + 4y - 3z = 22$
$-3x - 3y + 6z = -20$

28. $-3x + 8z = 14$
$5x - 2y + 3z = -8$
$-x - 4y - 2z = -2$

29. $-3x + z = 3$
$9x + 2y + 3z = -3$
$3x + y - 2z = 4$

30. $-x + 6y + 6z = -11$
$\frac{2}{3}y + 4z = 4$
$5x + 3y + 2z = 4$

31. $y + z = 0$
$-3x + 2y + 2z = 0$
$5x + 2y - 6z = -40$

32. $-5x + 5y + 4z = 14$
$3x + y - 4z = -10$
$x - 2y + 2z = 0$

33. $x - y + z = -14$
$-2x + 3y - 2z = 31$
$4x - 4y + 7z = -59$

34. $2x + z = 2$
$x + 3y - 2z = -3$
$-5x + 2y + 8z = -32$

35. $x - z = 5$
$-2x + 4y + 2w = 4$
$-2x - 2y - 3z - 2w = -6$
$2x - 5z + 5w = 21$

36. $2x - z + w = -2$
$3x + z + 4w = 7$
$4y - 2z + 5w = -9$
$y + 2z + 2w = 6$

37. $6x + 4y + z = 1$
$3x + y - 3w = -11$
$-4x + w = 0$
$-x + 4y = -9$

38. $-3z + 5w = -10$
$-x + z + 3w = -7$
$-3x - 3y + z = -6$
$-5x - y - 5z + 5w = -16$

39. $x + \frac{1}{2}y + 3z - 3w = -5$
$2x - \frac{3}{2}y + 3z + 5w = 16$
$-6x - w = -8$
$x - 6z = -1$

40. $4x + y - 5z = 14$
$x + y - 5z + 3w = \frac{25}{2}$
$-6x + 3y + 2z + 2w = -1$
$z - 3w = -2$

41. $3x + 2y - 3z - 3w = -3$
$-3x - y + 7z + 4w = -5$
$x - 3y - z + w = 7$
$4x + y + 4z = -12$

42. $-5x + y - z - 3w = 10$
$x - 3z = -11$
$x + 3y - z + w = -6$
$-3x - 3y + 5w = 1$

43. $-3x + 6y + 5z - 3w = 1$
$4x + 2z - 3w = 36$
$2x - 5y - 3z = 9$
$x + 2y + 5z - 4w = 29$

44. $-6x + 3y - 5w = -11$
$3x + 2y + 4w = 19$
$-3x - y + 2z + 3w = -4$
$y + 8z + w = -3$

45. $y + 8z + 2w = 6$
$3x + y + 4w = 11$
$x - 6y + 4z + 5w = 23$
$2x + 10y - 4z = -17$

46. $-4x + y + w = 16$
$x + y + 2z + 4w = 8$
$x - 4y + 2z - 4w = -21$
$x + 4y + 2z - 6w = -19$

Solve the following problems.

47. Kirchhoff's law, from circuit theory in electronics, states that the sum of the voltages around any loop of a circuit is 0. In a certain circuit with two loops, with i_1 the current in one loop and i_2 the current in the second loop, the application of Kirchhoff's law gives the system $\begin{array}{l} 60i_1 - 10i_2 = 116 \\ 10i_1 - 30i_2 = 8 \end{array}$. Solve for the currents i_1 and i_2.

48. For a certain electronics circuit Kirchhoff's law (problem 47) gives the system
$35i_1 + 12i_2 + 5i_3 = 197$
$60i_1 - 20i_2 - 10i_3 = 260$. Find i_1, i_2, and i_3.
$15i_1 + 10i_2 + 5i_3 = 95$

49. A certain scale is known to be very inaccurate for weights about 200 pounds. Three items, I_1, I_2, and I_3 must be weighed, and it is known that their weights are in the 200 pound range. Thus, the items are weighed together, and the following results are noted:
$I_1 + I_2 = 380$
$I_1 + I_3 = 390$. Find the weight of each of the three items.
$I_2 + I_3 = 410$

50. A chemical company stores a certain herbicide in two concentrations of herbicide and water: 6% solution and 15% solution. It needs to manufacture 1,000 gallons of a 10% solution by mixing the correct amounts of the two solutions. How many gallons of each should be mixed to obtain the required product?

$\boxed{51.}$ Certain amounts of a 20% and a 50% solution of alcohol are to be mixed to obtain 500 liters of a 30% solution. How many liters of each solution should be mixed together?

52. A certain mix of animal feed contains 10% protein; a second mix is 45% protein. How many pounds of each must be mixed to obtain 250 pounds of a 30% protein mix?

53. A trucking firm keeps a stock of 25% antifreeze solution and of 90% antifreeze solution on hand. How much of each should it mix to obtain 80 gallons of a 50% solution?

54. A paint manufacturer has two concentrations of its paint base on hand; one contains 4% linseed oil, the other contains 10% linseed oil. How many gallons of each should it mix to obtain 200 gallons of base of which 6.5% is linseed oil?

55. The following system of equations is found in a Chinese book from about 250 B.C.:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned}$$

It is solved by using steps similar to those shown in this section. Solve this system of equations.

Skill and review

1. Find the point of intersection of the two lines $y = 2x + 3$ and $2x - 3y = 6$.
2. Find $\log_3 81$.
3. Solve $9^{3x+1} = 27^x$.
4. Solve $\log(x - 1) - \log(x + 1) = 2$.
5. Solve $\log(x - 1) + \log(x + 1) = \log 2$.
6. Solve $x^3 - x^2 - x + 1 < 0$.

10–3 Systems of linear equations—Cramer's rule

Find the area of the triangle with vertices (−2,6), (3,−2), and (6,12).

Finding the area of a polygon such as that in this problem is one of the many types of problems that can be solved using systems of equations. Cramer's rule, studied in this section, provides another way to solve these systems.

Determinants

As defined in section 10–2 a square matrix of order n is an array (or matrix) of n rows and n columns. The word *order* implies the matrix in question is square. The general matrices of order 2 and order 3 are

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Associated with every square matrix is a real number called a **determinant.** The determinant[4] of an order 2 matrix is indicated by enclosing the matrix elements in vertical bars (as in the absolute value of a real number), and is defined as follows.

Determinants of an order 2 matrix

The determinant of a 2 × 2 matrix $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ is written $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$ and has the value $a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$.

Observe that the value is computed as the difference of the products of the elements on the two diagonals, with the diagonal from upper left to lower right used first.

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

■ ***Example 10–3 A***

Compute the determinant of $\begin{vmatrix} 2 & -4 \\ -3 & 5 \end{vmatrix}$

$$(2)(5) - (-3)(-4) = 10 - 12 = -2$$

The definition of the determinant of an order n matrix, $n > 2$, is defined in terms of the determinant of an order $n - 1$ matrix. This definition requires two additional definitions.

[4]The earliest determinant notations go back to the originator of determinants in Europe, Gottfried Wilhelm Leibniz (1693).

Minor of an element of a matrix

Given an nth order matrix A, the minor of element $a_{i,j}$ is the $(n - 1)$ order matrix formed by deleting the ith row and jth column of matrix A. We refer to this minor as $m_{i,j}$.

■ ***Example 10–3 B***

Find the minor $m_{2,3}$ of the matrix. This minor is the order 3 matrix found by deleting row 2 and column 3.

$$\begin{bmatrix} 4 & 0 & -1 & 3 \\ -2 & 3 & -5 & 6 \\ 0 & 5 & 1 & 2 \\ 0 & 8 & -3 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 4 & 0 & -1 & 3 \\ -2 & 3 & -5 & 6 \\ 0 & 5 & 1 & 2 \\ 0 & 8 & -3 & 7 \end{bmatrix} \text{ so } m_{2,3} = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 5 & 2 \\ 0 & 8 & 7 \end{bmatrix}$$

Sign matrix for an order n matrix

The sign matrix for an order n matrix is an order n matrix where each element has the value $(-1)^{i+j}$, where i is the row and j is the column. It looks like the following.

$$\begin{array}{c} \\ n \text{ rows} \end{array} \begin{array}{c} n \text{ columns} \\ \begin{bmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & \cdots & \\ + & - & + & - & \cdots & & \\ - & + & - & \cdots & & & \\ + & - & \cdots & & & & \\ - & \cdots & & & & & \\ \cdots & & & & & & \end{bmatrix} \end{array}$$

In the sign matrix, the upper left (i.e., 1,1) entry is $+$, and the pattern alternates with plus and minus signs through the rows and columns. The entry for a given row and column in the sign matrix can either be found by examination of the matrix itself or by computing $(-1)^{i+j}$, where i is the row and j is the column.

We are now prepared to define the determinant of matrices of orders greater than 2. It may look complicated, but the examples below will make it clear.

Determinant of an order n matrix

The determinant of an order n matrix, $n > 2$, is the sum of the products of each element of any row or column, $+1$ or -1, depending on the sign matrix and the determinant of its respective minor.

The following procedure reflects this definition.

Procedure for computing the determinant of an order *n* matrix.

1. Choose the row or column with the most zero elements.
2. For each nonzero element $a_{i,j}$ in this row or column compute the product of the following three factors:
   - $a_{i,j}$; the element itself
   - $(-1)^{i+j}$; the corresponding element in the sign matrix (+1 or −1)
   - $|m_{i,j}|$; the determinant of its minor
3. Add up all the values found in step 2.

Note
- In step 1 we can actually choose any row or column. Choosing the row or column with the most zeros minimizes the amount of work in the following steps.
- We generally perform steps 2 and 3 together. They are shown separately to help us understand them.

It has been proven that we get the same value for the determinant regardless of which row or column we choose for this "expansion."

The TI-81 calculator can compute determinants for matrices up to 6 by 6. This is illustrated in part 3 of the following example.

■ ***Example 10–3 C***

Compute each determinant.

1. $\begin{vmatrix} 0 & 5 & 2 \\ 3 & -2 & -1 \\ -3 & 4 & 6 \end{vmatrix}$

Step 1: No row or column has more than one zero. We choose row 1.

$$\begin{array}{cc} \text{Matrix} & \text{Sign Matrix} \\ \begin{vmatrix} 0 & \boxed{5} & \boxed{2} \\ 3 & -2 & -1 \\ -3 & 4 & 6 \end{vmatrix} & \begin{bmatrix} + & \boxed{-} & \boxed{+} \\ - & + & - \\ + & - & + \end{bmatrix} \end{array}$$

Steps 2 and 3: For each nonzero element in row 1 we form the product of the three factors shown in the steps.

$$-5\begin{vmatrix} 3 & -1 \\ -3 & 6 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ -3 & 4 \end{vmatrix} = 5(15) + 2(6) = -63$$

2. $\begin{vmatrix} 3 & 6 & 4 \\ 0 & 0 & -2 \\ -3 & -5 & 0 \end{vmatrix}$

Since row 2 has the most zeros we expand about row 2.

$$-(-2)\begin{vmatrix} 3 & 6 \\ -3 & -5 \end{vmatrix} = 2(3) = 6$$

3. $\begin{vmatrix} 2 & -1 & 3 & -4 \\ 4 & 3 & -1 & 2 \\ 1 & -5 & -2 & -3 \\ 1 & 5 & 2 & -6 \end{vmatrix}$

Expand around the first row. The calculation for one of the order 3 matrices is shown below.

$$= 2\begin{vmatrix} 3 & -1 & 2 \\ -5 & -2 & -3 \\ 5 & 2 & -6 \end{vmatrix} - (-1)\begin{vmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \\ 1 & 2 & -6 \end{vmatrix}$$

$$+ 3\begin{vmatrix} 4 & 3 & 2 \\ 1 & -5 & -3 \\ 1 & 5 & -6 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 3 & -1 \\ 1 & -5 & -2 \\ 1 & 5 & 2 \end{vmatrix}$$

$$= 2(99) + 77 + 3(209) + 4(-22)$$

$$= 814$$

The calculation of one of the order 3 determinants is shown.

$$\begin{vmatrix} 3 & -1 & 2 \\ -5 & -2 & -3 \\ 5 & 2 & -6 \end{vmatrix} = 3\begin{vmatrix} -2 & -3 \\ 2 & -6 \end{vmatrix} - (-1)\begin{vmatrix} -5 & -3 \\ 5 & -6 \end{vmatrix}$$

$$+ 2\begin{vmatrix} -5 & -2 \\ 5 & 2 \end{vmatrix} = 3(18) + 45 + 2(0) = 99$$

The determinant can be computed on the TI-81 as follows:

$\boxed{\text{MATRIX}}$ EDIT 1	Enter the values into matrix [A]
4 $\boxed{\text{ENTER}}$ 4 $\boxed{\text{ENTER}}$	This is a 4 × 4 matrix

Enter the values by row. Use the $\boxed{(-)}$ key for negative values. Use the $\boxed{\text{ENTER}}$ key after each value.

[MATRIX] 5	det
[A]	$\boxed{\text{2nd}}$ 1
$\boxed{\text{ENTER}}$	The result is 814

To view the matrix, enter $\boxed{\text{[A]}}$ $\boxed{\text{ENTER}}$.

Cramer's rule

It turns out that determinants can, among other things, be used to solve systems of linear equations by Cramer's rule.

> **Cramer's rule**
>
> Assume a given system of n linear equations in n variables. Let D represent the determinant of the coefficient matrix. Let D_x be the determinant of D with the x column replaced by the column of constants, D_y the determinant of D with the y column replaced by the column of constants, etc. Then,
>
> $$x = \frac{D_x}{D},\quad y = \frac{D_y}{D}, z = \frac{D_z}{D}, \text{etc.}$$

Example 10–3 D

Use Cramer's rule in each problem.

1. Solve the system $\begin{matrix} 3x - 2y = 26 \\ 3y = 5 \end{matrix}$.

$$D = \begin{vmatrix} 3 & -2 \\ 0 & 3 \end{vmatrix} \quad \text{The coefficients from } \begin{matrix} 3x - 2y \\ 0x + 3y \end{matrix}$$

$$= 9 - 0 = 9$$

$$D_x = \begin{vmatrix} 26 & -2 \\ 5 & 3 \end{vmatrix} \quad \text{Replace the } x \text{ column in } D \text{ by the coefficients } \begin{matrix} 26 \\ 5 \end{matrix}$$

$$= 78 - (-10) = 88$$

$$D_y = \begin{vmatrix} 3 & 26 \\ 0 & 5 \end{vmatrix} \quad \text{Replace the } y \text{ column in } D \text{ by the coefficients } \begin{matrix} 26 \\ 5 \end{matrix}$$

$$= 15 - 0 = 15$$

So by Cramer's rule, $x = \dfrac{D_x}{D} = \dfrac{88}{9}$ and $y = \dfrac{D_y}{D} = \dfrac{15}{9} = \dfrac{5}{3}$.

2. Solve the system.

$2x - y = 5$
$x + 3z = 0$
$y - 2z + w = -2$
$x - w = 3$

$$D = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & -1 \end{vmatrix} = 11$$

$$D_x = \begin{vmatrix} 5 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -2 & 1 & -2 & 1 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 18; \quad D_y = \begin{vmatrix} 2 & 5 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & -2 & -2 & 1 \\ 1 & 3 & 0 & -1 \end{vmatrix} = -19$$

$$D_z = \begin{vmatrix} 2 & -1 & 5 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 3 & -1 \end{vmatrix} = -6; \quad D_w = \begin{vmatrix} 2 & -1 & 0 & 5 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & -2 \\ 1 & 0 & 0 & 3 \end{vmatrix} = -15$$

Thus, $x = \frac{18}{11}$, $y = -\frac{19}{11}$, $z = -\frac{6}{11}$, and $w = -\frac{15}{11}$.

Neither of the systems illustrated in example 10–3 D are inconsistent or dependent. *When the determinant of the coefficient matrix is zero ($D = 0$), the system is either inconsistent or dependent.* If any of the other determinants D_x, D_y, D_z, are not zero the system is inconsistent; if they are all zero, the system is dependent.

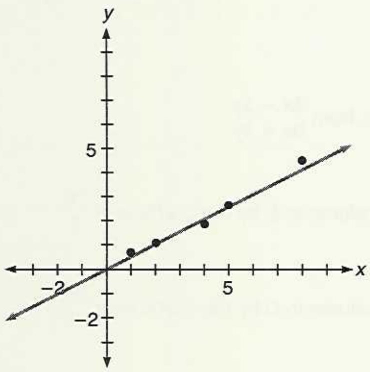


Figure 10–2

Linear regression

In the modeling of many situations it is desirable to find the equation $y = mx + b$ of a straight line that best fits a set of measured data. For example, in figure 10–2 we see the graph of a line that closely fits the points (1,0.7), (2,1.1), (4,1.8), (5,2.6), and (8,4.4).

This line is called the **least-squares line,** and the process of finding the linear is called **linear regression.** We present a method for finding the values m and b for the equation $y = mx + b$ of the least-squares line.

Let

Y equal the sum of the y data.

X equal the sum of the x data.

P equal the sum of the products of the x and y in each observation.

S equal the sum of the squares of the x data.

N equal the number of observations.

Then it can be shown that

$$Xm + Nb = Y$$
$$Sm + Xb = P$$

■ *Example 10–3 E*

Find the least-squares line for the data: (1,0.7), (2,1.1), (4,1.8), (5,2.6), and (8,4.4). The computations for the example data are most easily done using a table (table 10–1).

	x	y	xy	x^2
	1	0.7	0.7	1
	2	1.1	2.2	4
	4	1.8	7.2	16
	5	2.6	13.0	25
	8	4.4	35.2	64
Totals	20	10.6	58.3	110
	X	Y	P	S

Table 10–1

Also, $N = 5$, the number of observations. Using the values from the table our equations become

$$\begin{aligned} Xm + Nb &= Y \qquad [1]\ 20m + 5b = 10.6 \\ Sm + Xb &= P \qquad [2]\ 110m + 20b = 58.3 \end{aligned}$$

Using Cramer's rule we compute m and b:

$$D = \begin{vmatrix} 20 & 5 \\ 110 & 20 \end{vmatrix} = -150$$

$$D_m = \begin{vmatrix} 10.6 & 5 \\ 58.3 & 20 \end{vmatrix} = -79.5$$

$$D_b = \begin{vmatrix} 20 & 10.6 \\ 110 & 58.3 \end{vmatrix} = 0$$

$$m = \frac{D_m}{D} = \frac{-79.5}{-150} = 0.53 \qquad b = \frac{D_b}{D} = \frac{0}{-150} = 0$$

Thus, the least-squares line $y = mx + b$ is $y = 0.53x$. ■

Calculators and computers can be used to find the values m and b for the least-squares line. The procedure is the same as that shown in example 3–2 D (chapter 3) for two points, except that more than two points are being entered.

Mastery points

Can you

- Find the determinant of matrices of order 2 and above?
- Use Cramer's rule to solve systems of linear equations?

Exercise 10–3

Compute the determinant of the following matrices.

1. $\begin{bmatrix} 1 & -4 \\ -3 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 6 \\ -\frac{1}{2} & 3 \end{bmatrix}$ 3. $\begin{bmatrix} -7 & \frac{3}{4} \\ \frac{2}{3} & 6 \end{bmatrix}$ **4.** $\begin{bmatrix} -3 & -4 \\ 6 & 8 \end{bmatrix}$

5. $\begin{bmatrix} -3\pi & -4\pi \\ 2 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} \sqrt{2} & -\sqrt{8} \\ 3 & -2 \end{bmatrix}$ **7.** $\begin{bmatrix} 4 & 0 & -5 \\ 1 & 5 & -2 \\ 0 & 3 & 7 \end{bmatrix}$ **8.** $\begin{bmatrix} 2 & -1 & 2 \\ 4 & -3 & 2 \\ -6 & 1 & 3 \end{bmatrix}$

$\boxed{9.}$ $\begin{bmatrix} 2 & \frac{2}{3} & -1 \\ 4 & -1 & \frac{1}{2} \\ -3 & 0 & -2 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & -2 & -3 \\ 1 & 5 & -2 \\ 0 & 3 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 4 & 3 & -2 \\ -1 & 0 & 2 \\ -2 & 1 & 4 \end{bmatrix}$ **12.** $\begin{bmatrix} 2 & -1 & 0 \\ 4 & -1 & 2 \\ -3 & 1 & -2 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 1 & 3 \\ 1 & -2 & 5 \\ 0 & 0 & 7 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ **15.** $\begin{bmatrix} -3 & 6 & 0 \\ 4 & -1 & 0 \\ -3 & 1 & -5 \end{bmatrix}$ **16.** $\begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{3} \\ -4 & 2 & 2 \\ 0 & \frac{3}{4} & -3 \end{bmatrix}$

17. $\begin{bmatrix} \sqrt{2} & 0 & 3 \\ \sqrt{8} & -5 & 2 \\ -\sqrt{2} & -1 & 7 \end{bmatrix}$ **18.** $\begin{bmatrix} 4 & -8 & 3 \\ 0 & 5 & -12 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ **19.** $\begin{bmatrix} 4 & 0 & -5 & 1 \\ 5 & -2 & 0 & 3 \\ 7 & 0 & 1 & 0 \\ 4 & -2 & 0 & 3 \end{bmatrix}$ **20.** $\begin{bmatrix} 3 & -2 & -1 & 1 \\ -5 & -2 & 2 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & -6 & 0 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 0 & -3 & 2 & 0 \\ -2 & 5 & 10 & 0 \\ -4 & 3 & 0 & 1 \\ 2 & -3 & 1 & 0 \end{bmatrix}$ **22.** $\begin{bmatrix} 4 & 0 & -2 & 1 \\ 2 & -2 & 0 & 3 \\ 1 & 0 & 1 & 3 \\ 4 & -2 & 0 & 3 \end{bmatrix}$ $\boxed{23.}$ $\begin{bmatrix} 4 & 5 & 1 & 0 \\ -2 & 1 & 3 & 7 \\ 0 & 1 & 2 & 0 \\ 4 & -2 & 0 & 3 \end{bmatrix}$ **24.** $\begin{bmatrix} 3 & -2 & -1 & 0 \\ -5 & -1 & 2 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & -2 & 0 & 3 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 2 & -4 & 0 \\ -2 & 5 & 6 & 0 \\ 0 & 3 & 0 & 5 \\ 2 & -3 & 1 & 0 \end{bmatrix}$ **26.** $\begin{bmatrix} -2 & 1 & -2 & 3 \\ 2 & -4 & 0 & 3 \\ 1 & 2 & 1 & 3 \\ 4 & -2 & 0 & 3 \end{bmatrix}$ **27.** $\begin{bmatrix} 3 & 0 & 2 & 1 \\ 5 & 0 & 0 & 3 \\ 5 & 0 & 1 & -3 \\ 4 & -2 & 2 & 3 \end{bmatrix}$ **28.** $\begin{bmatrix} 3 & 0 & -1 & 1 \\ -5 & -2 & 0 & 3 \\ -3 & 2 & -1 & 4 \\ 0 & -6 & 0 & 7 \end{bmatrix}$

Solve the following systems of equations by Cramer's rule; if the system is dependent or inconsistent state that.

29. $\begin{aligned} -3x - 4y &= 0 \\ -x + 9y &= 4 \end{aligned}$ **30.** $\begin{aligned} -9y &= 1 \\ -4x - 6y &= -1 \end{aligned}$ **31.** $\begin{aligned} 9y &= -5 \\ -8x - 8y &= 12 \end{aligned}$ **32.** $\begin{aligned} 3x + 2y &= 12 \\ -8x - 2y &= 2 \end{aligned}$

33. $\begin{aligned} -2x - 5y &= 9 \\ -7x - 8y &= -4 \end{aligned}$ **34.** $\begin{aligned} -7x + 6y &= -6 \\ -10x + 3y &= 3 \end{aligned}$ **35.** $\begin{aligned} 6x + 8y + 3z &= -4 \\ x - 3y &= -1 \\ 5x + 9y + 7z &= -6 \end{aligned}$ **36.** $\begin{aligned} -5x - 6y - 3z &= 5 \\ -5x + 5y &= 7 \\ x + 8y - 3z &= 9 \end{aligned}$

$\boxed{37.}$ $\begin{aligned} 9x - 3y - 3z &= -6 \\ x - 4y - 6z &= 7 \\ 3x + 2y &= -3 \end{aligned}$ **38.** $\begin{aligned} 7y + 5z &= 5 \\ 9x - 2y + 9z &= 8 \\ 8x + 2z &= 9 \end{aligned}$ **39.** $\begin{aligned} -x + 2y &= -2 \\ x + 5y - 2z &= 3 \\ 4x + 2y &= 6 \end{aligned}$ **40.** $\begin{aligned} -4x + 3y + 3z &= 9 \\ 5x - 4z &= 3 \\ -x - 6y + 2z &= -5 \end{aligned}$

41. $\begin{aligned} x - y &= 4 \\ -4x + 9y - 3z &= -3 \\ -3x - y &= 7 \end{aligned}$ **42.** $\begin{aligned} 8x - 6y + 8z &= -5 \\ -5x + 4y + 6z &= -2 \\ 9x + 3y + 7z &= -6 \end{aligned}$ 43. $\begin{aligned} 6x + 8y &= -3 \\ 4x - 5y - 2z &= 0 \\ 10x + 3y - 2z &= 2 \end{aligned}$ **44.** $\begin{aligned} 3x - z &= 6 \\ 2y + 3z &= -8 \\ y &= 5 \end{aligned}$

45. $2x + 2y + 3z - 2w = 2$
$-x + 2y + 5z - w = -3$
$-4x + 4y - 2z - 4w = -2$
$-4x + 4y - 4w = 2$

46. $-4x + 5y - 3z + 5w = 2$
$6x - 2y + 6z = 2$
$4y + 6z - 4w = 2$
$4x - 2y - 2w = -1$

47. $x - y + 4z - 4w = 4$
$3x + 4y + 2z = -1$
$5x - 2z + 6w = 6$
$-2x + 4y - 2z + w = 2$

48. $y - w = -2$
$-y + 4z - 3w = -1$
$-x - 3z - 4w = 1$
$4x + 5y + 2z + 6w = 3$

49. $2x - y + 6z - 4w = 1$
$5x - y + 2z + 4w = 4$
$2x - 3y + 4z = 5$
$-3y + 4z - 4w = 5$

50. $3x + 4y - 2w = 0$
$2y - z - w = 3$
$z + w = -2$
$x + z = 2$

51. Solve for the variable C in the following system:
$2A - B + 3C - D = 5$
$A + B - 2D + E = 0$
$-B - C = 10$
$3A - C + D + E = -4$
$C - E = -20$

52. Solve for E in the system of problem 51.

In problems 53 through 56 use Cramer's rule to compute the least-squares line for each set of data; compute the values of m and b to two decimal places. Use the formulas given in example 10–3 E.

53. (0,0.5), (1,1.8), (2,3.0), (4,5.0), (6,7.6)

54. (1,−2.0), (3,−7.8), (4,−11.5), (5,−14.4)

55. (1.5,−4.8), (2,−4.0), (3,−2.0), (4.5,1.4), (6,4.7), (6.5,5.6)

56. (0,−64), (1,−56), (2,−40), (3,−35), (4,−20), (5,−10)

57. A company is studying failure rates in its line of power steering pumps for automobiles. It has measured the following data, where the first element of each ordered pair is the age of the pump in years, and the second is percentage of failures for pumps of that age:

(1,3), (2,5), (3,6), (4,9)

For example, at 2 years old, 5% of the pumps fail. Find the least-squares line for these data and use it to predict the percentage of pump failures, to the nearest tenth of a percentage, in the fifth and sixth years.

58. Use Cramer's rule to derive a formula, in terms of X, Y, P, N, and S, which will compute m and b directly. That is, solve the system $\begin{matrix} Xm + Nb = Y \\ Sm + Xb = P \end{matrix}$ for m and for b.

59. The data in the table represents the world's record for the 1-mile run in the year shown; use it to (a) find its least-squares line and then (b) predict the year when a 3 minute 35 second mile will be run. To make things easier, rewrite the year 1875 as 0, 1895 as 20, etc., subtracting 1875 from each year. Also, describe the time in seconds, using 4:24.5 as the base. For example, 4:17 − 4:24.5 = −7.5 (seconds). Thus, for example the data pair 1895, 4:17 is the ordered pair (20,−7.5).

Year	Time
1875	4:24.5
1895	4:17
1915	4:12.6
1923	4:10.4
1934	4:06.8
1945	4:01.4
1954	3:59.4
1965	3:53.6
1975	3:49.4

If the three points that form the vertices of a triangle are (x_1,y_1), (x_2,y_2), and (x_3,y_3), then it can be shown that the area of the triangle is the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Problems 60 through 63 refer to this fact.

60. Find the area of the triangle with vertices (0,2), (5,3), and (2,8).

61. Find the area of the triangle with vertices (−2,6), (3,−2), and (6,12).

62. Describe the set of all points (x,y) that form the third vertex of a triangle with vertices at (1,3) and (5,1) and area 10.

63. Find the area of the quadrilateral (four-sided figure) with vertices (0,2), (3,0), (5,8), and (1,10).

64. Show that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ produces the equation of the straight line through the distinct points (x_1,y_1) and (x_2,y_2).

65. Find the equation of the straight line that passes through the points $(2,-3)$ and $(5,4)$. See problem 64.

66. Find the equation of the straight line with x-intercept -4 and y-intercept 2. See problem 64.

67. A parabola is the graph of an equation of the form $y = ax^2 + bx + c$. State the equation of the parabola that will pass through the points $(-4,2)$, $(1,3)$, and $(2,8)$.

68. Find the equation of the parabola that passes through the points $(-1,0)$, $(1,3)$ and $(3,10)$. See problem 67.

69. Problem 64 showed one way to find the equation of a straight line that passes through two points; another method is to realize that a nonvertical straight line is the graph of an equation of the form $y = mx + b$. Two points uniquely determine a straight line. Find the equation of the straight line that passes through the points $(5,-1)$ and $(8,6)$ by substituting these values into the equation $y = mx + b$ and thereby obtaining a system of two equations in two unknowns.

70. Find the equation of the straight line that passes through the points $(-2,8)$ and $(5,1)$. Refer to problem 69.

71. A straight line passes through the points $(-2,-1)$ and $(3,2)$; a second straight line passes through the points $(-6,2)$ and $(5,-7)$. Find the point at which these two straight lines intersect.

72. Kirchhoff's law, from circuit theory in electronics, states that the sum of the voltages around any loop of a circuit is 0. In a certain circuit with two loops, with i_1 the current in one loop and i_2 the current in the second loop, the application of Kirchhoff's law gives the system $20i_1 - 10i_2 = 40$ and $10i_1 - 4i_2 = 25$. Solve for the currents i_1 and i_2.

73. For a certain electronics circuit Kirchhoff's law (problem 72) gives the following system. Find the currents i_1, i_2, and i_3.

$$\begin{aligned} 35i_1 + 12i_2 + 5i_3 &= 50 \\ 30i_1 - 20i_2 - 10i_3 &= -40 \\ 15i_1 + 10i_2 + 5i_3 &= 60 \end{aligned}$$

74. A certain scale is known to be very inaccurate for weights about 200 pounds. Three items I_1, I_2, and I_3 must be weighed, and it is known that their weights are in the 200 pound range. Thus, the items are weighed together, and the following results are noted. Find the weight of each of the three items.

$$\begin{aligned} I_1 + I_2 &= 370 \\ I_1 + I_3 &= 395 \\ I_2 + I_3 &= 415 \end{aligned}$$

75. An inheritance of \$36,000 is to be given to three charities, x, y, and z, in the ratios 3:4:5. How much will each charity get?

$$\left(\textit{Hint: } x + y + z = 36{,}000,\ \frac{x}{3} = \frac{y}{4} = \frac{z}{5}.\right)$$

76. Divide \$84,000 three ways so the ratios of each amount are 5:6:10. See problem 75.

77. A problem with finding approximate solutions to systems of equations is determining when a solution is "correct." For example, consider the system

$$\begin{aligned} 0.12658x + 0.25315y &= 0.37973 \\ 0.88606x + 1.77213y &= 2.65819 \end{aligned}$$

The "solution" (3,0) gives an error of only -0.00001 in the first equation and 0.00001 in the second, yet this solution is actually not too close to the "true" solution. Find the true solution using Cramer's rule.

78. A problem similar to problem 77 occurs when evaluating determinants. Compute the determinant of each of the following matrices, and compare the results:

$$\begin{bmatrix} 11 & 19 & 9 \\ 25 & 48 & 24 \\ -124 & 12 & 65 \end{bmatrix} \begin{bmatrix} 11.01 & 19 & 9 \\ 25 & 48 & 24 \\ -124 & 12 & 65 \end{bmatrix}$$

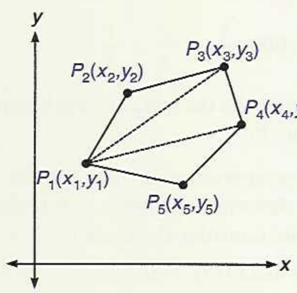
79. In the text we defined the determinant of a matrix of order greater than 2 in a different fashion than that for an order 2 matrix. Also, we did not define the determinant of an order 1 matrix. Find a definition for the determinant of an order 1 matrix that would then allow us to find the determinant of an order 2 matrix using the definition for matrices of order greater than 2.

80. In problems 60–63 we stated a formula which gives the area of a triangle with vertices at points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, and $P_3(x_3,y_3)$. Show that this formula is equivalent to the formula
$\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$.

81. Consider the figure shown. It shows how a polygon can be divided up into triangles. The area of the polygon is the sum of the areas of the triangles.

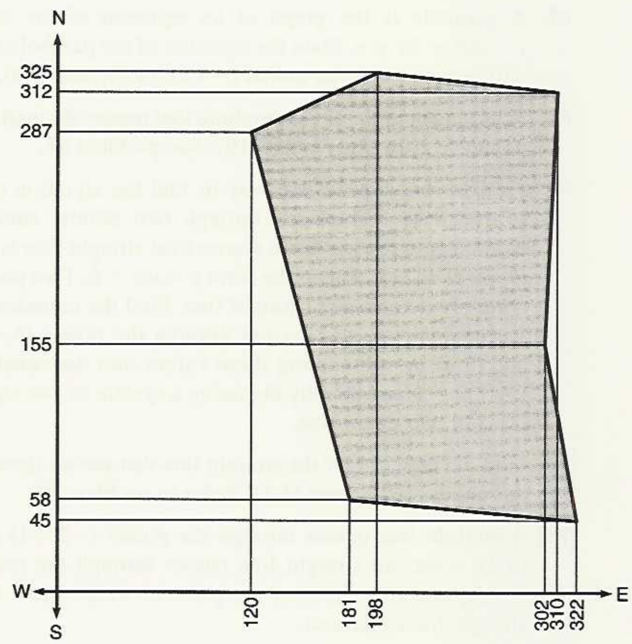
a. Use the determinant of problem 80 to show that the following is a formula for the area of a four-sided polygon (a quadrilateral):
$\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$.

b. Similarly, show that the area of a five-sided polygon is given by the formula:
$\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_5 - x_5y_4) + (x_5y_1 - x_1y_5)]$.



82. The figure shows a piece of land whose area is to be found. A north-south base line and an east-west base line are laid out and a survey made as shown. The distances shown are in meters from the origin. Find the area to the nearest square meter.

Do this as follows. Establish a formula for the area of a polygon of six sides. Use problem 81 as a guide for doing this. Then use the measurements shown to establish coordinates for each vertex of the piece of land and apply the formula.



Skill and review

1. Solve $2x - 3 < 8$.
2. Is the point $(2,-1)$ a solution to the statement $2x + y > 2$?
3. Graph the lines $y = 2x - 1$ and $y = -\frac{1}{3}x + 1$ in the same graph. Label the point where the lines intersect.
4. Add $7\frac{1}{2}\%$ of \$1,200 to 5% of \$1,800.
5. Solve $3(2x + 3) - 2(5x + 3) = x$.
6. Solve $\frac{4x - 1}{x} < -5x$.
7. Solve $\log^2 x - \log x = 6$.
8. Graph $f(x) = \log_2(x + 1)$.
9. Graph $f(x) = x^2 + 3x - 4$.

10–4 Systems of linear inequalities

A certain animal food is available from two sources, A and B. A supplies 5 grams per pound (5 gm/lb) of protein and 4 gm/lb of carbohydrates. B supplies 8 gm/lb and 3 gm/lb of protein and carbohydrates. A costs 15 cents per pound, and B is 18 cents. If the *minimum* daily requirement for this animal is 20 gm of protein and 12 gm of carbohydrates, how much of A and B should be used to minimize costs?

This problem can be solved by a method called linear programming, which we will investigate after examining linear inequalities in two variables.

In the previous sections we have focused our attention on systems of linear equations; we now turn our attention to systems of linear inequalities in two variables. The solution sets in these cases are best indicated by their graphs. We will see that a very important application of these systems is linear programming, mentioned above, which is used extensively in the discipline called operations research.

Linear inequalities in two variables

A linear inequality in two variables would be, for example, $2x - 3y < 6$. We are interested in describing all points (x,y) that make this inequality true. We know that the corresponding equality $2x - 3y = 6$ is a straight line.

Table 10–2 shows various ordered pairs, plotted in figure 10–3, along with the statement ''true'' or ''false'' depending on the truth value of $2x - 3y < 6$ for that ordered pair (x,y). For example, if $(x,y) = (3,-2)$ (point A in the figure), we calculate

$2x - 3y < 6$

$2(3) - 3(-2) < 6$ Replace x with 3 and y with -2

$12 < 6$, which is false.

Point	$2x - 3y < 6$	True/False
$A\ (3,-2)$	$12 < 6$	False
$B\ (-1,-3)$	$7 < 6$	False
$C\ (5,1)$	$7 < 6$	False
$D\ (-1,-1)$	$1 < 6$	True
$E\ (-1,2)$	$-8 < 6$	True
$F\ (4,3)$	$-1 < 6$	True

Table 10–2

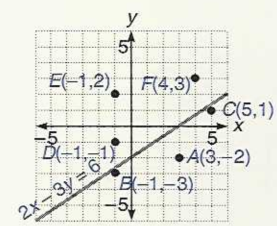


Figure 10–3

The points for which the inequality is true are circled in the figure. All of them are above the line $2x - 3y = 6$ (D,E,F), while those below the line (A, B, and C) all make it false. This might lead us to guess that any point above the line will make the inequality true, and any point below will make it false. It can be proven that this guess is correct. A straight line divides the plane up into two halves, and one of these two half-planes is the solution to a corresponding strict linear inequality in two variables.

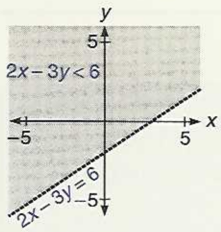


Figure 10–4

We could indicate the solution set to the linear inequality $2x - 3y < 6$ by shading in the half-plane above the line $2x - 3y = 6$. This is illustrated in figure 10–4. Observe also that we draw the line $2x - 3y = 6$ as a dashed line. This is to indicate that it is not part of the solution; any point (x,y) on this line satisfies $2x - 3y = 6$, not $2x - 3y < 6$.

If the inequality were the weak[5] inequality $2x - 3y \leq 6$, we would show the line as a solid line since the line would be part of the solution.

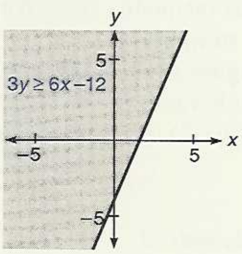
To graph a linear inequality in two variables

- Graph the corresponding linear equality (the straight line). If the inequality is a weak inequality draw the straight line as a solid line, otherwise a dashed line.
- Try a test point from one of the half-planes in the inequality. If this point makes the inequality true then that half-plane is the solution set, so shade in that half-plane; otherwise shade in the other half-plane.

Example 10–4 A illustrates this procedure.

■ *Example 10–4 A*

Graph the solution set of each inequality.



1. $3y \geq 6x - 12$

Graph $3y = 6x - 12$ as a solid line since this is a weak inequality. Do this by plotting the intercepts and drawing the line through them.

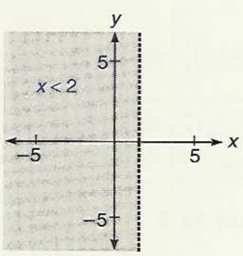
$3y > 6x - 12$

$3(0) > 6(0) - 12$ Try the test point (0,0) in the inequality

$0 > -12$ True

Since (0,0) satisfies the inequality the half-plane which includes the origin (0,0) is in the solution set. Shade in this half-plane. The solution set is the half-plane along with the line $3y = 6x - 12$.

Note In a weak inequality the line itself is part of the solution set.



2. $x < 2$

Graph $x = 2$ as a (vertical) dashed line.

$x < 2$ Now determine which half-plane is the solution

$0 < 2$ Try (0,0) in the inequality; this is true

Thus, the solution set is the half-plane containing the origin. Shade in this half-plane. ■

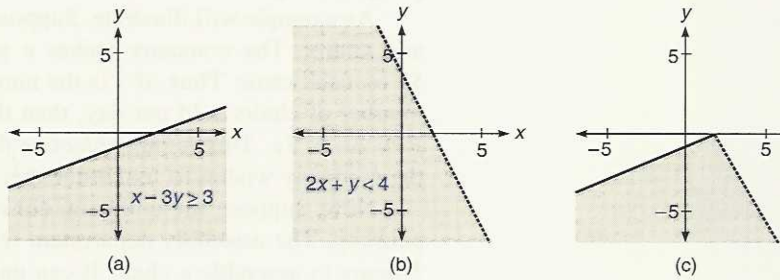
Systems of linear inequalities in two variables

The graph of a *system* of linear inequalities in two variables consists of the intersection of the solution sets of each inequality; graphically, this means where the graphs of all the inequalities overlap.

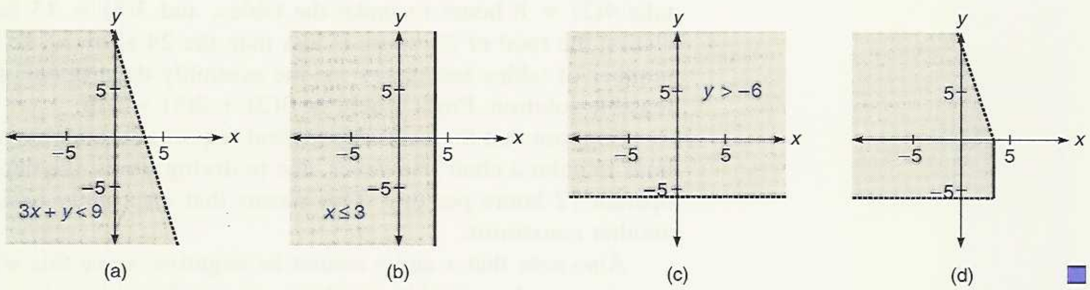
[5]Recall that a weak inequality is represented by $\geq$ or $\leq$ (section 1–1).

■ ***Example 10–4 B*** Graph the solution set to each system of inequalities.

1. $x - 3y \geq 3$
$2x + y < 4$
Part (a) of the figure shows the solution to the inequality $x - 3y \geq 3$. Part (b) shows the solution to $2x + y < 4$. Part (c) shows the solution set. Observe the solid line, which is part of the solution set. This is where the line $x - 3y = 3$ intersects the inequality $2x + y < 4$.



2. $3x + y < 9$
$x \leq 3, y > -6$
The solutions to each of these three inequalities are shown in parts (a), (b), and (c) of the figure, and the final answer in part (d).



Linear programming

Linear programming is a mathematical tool used by organizations to maximize or minimize chosen values. For example, it might be used to maximize profits or minimize costs in a company. We will show only examples that employ two variables, but note that linear programming, developed in the 1940s for the Air Force by the mathematician George Dantzig, is used in more complicated situations every day in many applications. Students will encounter this subject in more depth in a course on finite mathematics; courses in linear programming and operations research would present the full power of this topic.

A **linear programming problem** in two variables is a problem that can be described by a set of linear inequalities, called **constraints,** and a linear equation in two variables, called the **objective function.** The constraints form a set of **feasible solutions.** The set of linear equalities that correspond to the set of inequalities form a boundary, and it can be proven that *the maximum or minimum value of the objective function can be found at one of the vertices of this boundary.* This is called the **fundamental principle of linear progamming.**

An example will illustrate. Suppose a company makes two products, tables and chairs. The company makes a profit of \$3 on each table it sells and \$2 on each chair. Thus, if x is the number of tables sold per day, and y is the number of chairs sold per day, then the day's profit P could be described as $P = 3x + 2y$. This is our *objective function* for this problem, and naturally the company wishes to maximize this value.

Now, suppose the company consists of two departments, assembly and painting. The assembly department requires 4 hours to assemble a table and 3 hours to assemble a chair. It can only assemble one table or one chair at a time. Now, $4x$ is the time required to assemble x tables at 4 hours per table, and $3y$ is the time required to assemble y chairs at 3 hours per chair. The total time cannot exceed the 24 hours in one day. The department operates 24 hours per day, so it is limited by the *constraint* $4x + 3y \leq 24$.

For example, if two tables and five chairs are made in one day, it would take $4(2) = 8$ hours to make the tables, and $3(5) = 15$ hours to make the chairs; the total of 23 hours is less than the 24 allowed, so this is a possible number of tables and chairs for the assembly department, and is therefore a feasible solution. Profit would be $3(2) + 2(5) = \$16$.

Suppose that the paint department requires 3 hours to paint a table, and 1 hour to paint a chair. However, due to drying times, the department can only operate 12 hours per day. This means that on a daily basis, $3x + y \leq 12$, another *constraint.*

Also note that x and y cannot be negative, since this would mean that a negative number of tables or chairs was produced in a day.

We can now formulate our linear programming problem for this company as follows: given the constraints

$$4x + 3y \leq 24$$
$$3x + y \leq 12$$
$$x \geq 0, y \geq 0$$

maximize the objective function $P = 3x + 2y$.

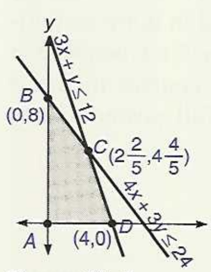


Figure 10–5

The set of constraints is graphed in figure 10–5. The point of intersection C of the first two constraints is found by any of the methods shown in sections 10–1 through 10–3. The line segments connecting the points A, B, C, and D form a polygon, which is shaded in. This shaded area defines what is called the *set of feasible solutions;* our goal is to find the feasible solution that gives the largest value for profit P. The *maximum* value of P can be found *at one of the vertices A, B, C,* or *D.* Table 10–3 shows the value of P for each of these points.

	Point	$P\ (= 3x + 2y)$
A	(0,0)	0
B	(0,8)	16
C	$(2\frac{2}{5}, 4\frac{4}{5})$	16.8
D	(4,0)	12

Table 10–3

We can see that the maximum value of P is \$16.80 per day, corresponding to a production of 2.4 tables and 4.8 chairs per day. This solution assumes that it is possible to make fractional parts of tables and chairs in a day.

We can see why the solution is one of the vertices if we consider the graph of the objective function $P = 3x + 2y$, or $y = -\dfrac{3}{2}x + \dfrac{P}{2}$. For differing values of P, this is a family of parallel lines (figure 10–6). As P increases, the lines move "up." When the lines move past a point, they no longer intersect the set of feasible solutions. It can be seen that this point will be a vertex.

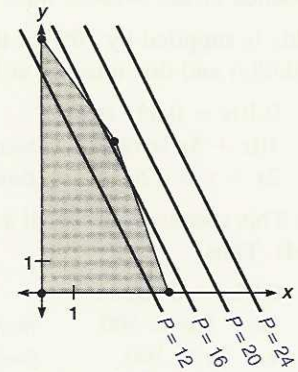


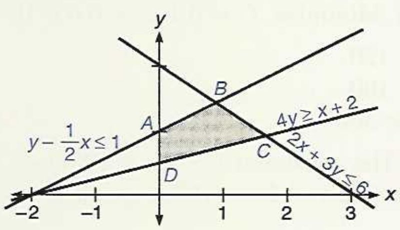
Figure 10–6

Note If the line corresponding to the objective function is parallel to an edge, there may in fact be many solutions, any of which maximizes the objective function.

■ Example 10–4 C

1. Maximize the value of the objective function, $Z = x + 3y$, with regard to the constraints indicated.

$$y - \tfrac{1}{2}x \le 1$$
$$4y \ge 2 + x$$
$$2x + 3y \le 6$$
$$x \ge 0,\ y \ge 0$$



The graph of the feasible solutions is shown in the figure. The points that form the vertices of the polygon that is the set of feasible solutions are labeled A, B, C, and D. Point B is the intersection of the lines $y - \frac{1}{2}x = 1$ and $2x + 3y = 6$. Point C is the intersection of the lines $4y = x + 2$ and $2x + 3y = 6$.

Point	$Z = x + 3y$
$A\ (0,1)$	3
$B = (\frac{6}{7}, \frac{10}{7})$	$5\frac{1}{7}$
$C\ (\frac{18}{11}, \frac{10}{11})$	$4\frac{4}{11}$
$D\ (0, \frac{1}{2})$	$1\frac{1}{2}$

The coordinate of these points and the value of $Z = x + 3y$ at these points is shown in the table where we see that $Z = 5\frac{1}{7}$, at point B, is the maximum value of Z. Thus, the solution is the point $B(\frac{6}{7}, 1\frac{3}{7})$; at this point $Z = 5\frac{1}{7}$.

2. Two fertilizers are available; one is a 10-5-10 mix and the second is 5-10-25. The first number refers to percentage of nitrogen, the second to percentage of phosphorus, and the third to percentage of potash. The first fertilizer sells for \$0.10 per pound, and the second for \$0.06 per pound. A farmer must put the following minimum amounts of each nutrient on a field: 6 pounds of nitrogen, 5 pounds of phosphorus, and 15 pounds of potash. How much of each fertilizer should the farmer use to minimize cost?

Let x be the number of pounds of the 10-5-10 mix, and let y be the number of pounds of the 5-10-25 mix. Now we consider each nutrient.

Nitrogen: This is supplied by 10% of the first fertilizer ($0.10x$) and 5% of the second ($0.05y$) and this must be at least 6 pounds. Thus,

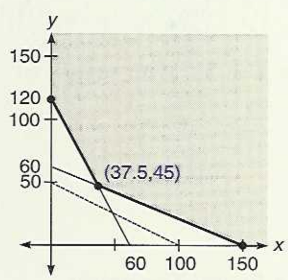
$$0.10x + 0.05y \geq 6$$
$$10x + 5y \geq 600 \quad \text{Multiply each member by 100}$$
$$2x + y \geq 120 \quad \text{Divide each member by 5}$$

Phosphorus: This comes from 5% of x and 10% of y, and must be at least 5 pounds. Thus,

$$0.05x + 0.10y \geq 5$$
$$5x + 10y \geq 500 \quad \text{Multiply each member by 100}$$
$$x + 2y \geq 100 \quad \text{Divide each member by 5}$$

Potash: The minimum amount of 15 pounds comes from 10% of x and 25% of y, so,

$$0.10x + 0.25y \geq 15$$
$$10x + 25y \geq 1500$$
$$2x + 5y \geq 300$$



Point	$C = 0.10x + 0.06y$
(0,120)	7.20
(150,0)	15.00
(37.5,45)	6.45

The cost function C is $C = 0.10x + 0.06y$. Thus, we need to solve the following linear programming problem. Minimize, $C = 0.10x + 0.06y$ if

$$2x + y \geq 120$$
$$x + 2y \geq 100$$
$$2x + 5y \geq 300$$

We graph the set of feasible solutions. The constraint $x + 2y \geq 100$ does not provide any feasible solutions. (The line $x + 2y = 100$ is graphed as a dashed line.)

We must check the points at the vertices (0,120), (150,0), and (37.5,45). This is shown in the table. We can see that the cost is minimized by using 37.5 pounds of the first fertilizer and 45 pounds of the second. The cost will be \$6.45. ■

Mastery points

Can you

- Graph the solution to a linear inequality in two variables?
- Graph the solution to a system of linear inequalities in two variables?
- Solve a linear programming problem that is defined in two variables?
- Convert certain problems into a linear programming problem in two variables and solve that problem?

Exercise 10–4

Graph the solution set of the following linear inequalities.

1. $3x - 2y > 6$ **2.** $x + 3y < 9$ **3.** $5x > y - 1$ **4.** $2x + 6y \geq 18$ **5.** $9 - 2x > 3y$
6. $3y - x < 0$ **7.** $x + 2y \geq -4$ **8.** $-x - 4 < y$ **9.** $6 \geq x + 7y$ **10.** $-9x + 3 > y$
11. $15y < 12x + 5$ **12.** $x - 8 \leq \dfrac{y}{2}$ **13.** $x > 2$ **14.** $y < -1$ **15.** $2x \leq 5$
16. $\frac{5}{3}y - \frac{7}{2} < 3x$ **17.** $\frac{5}{6}x - 3y \geq 9$ **18.** $y - \dfrac{x}{5} < \dfrac{3}{10}$ **19.** $0.3x - 1.2y \leq 4$ **20.** $1.2y \leq 2x - 0.5$
21. $1.5 > x + 0.1y$

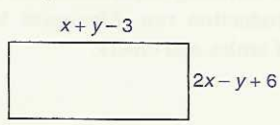
22. The dimensions in yards of a piece of land are length x and width y. The length is to be increased by 5 yards, while the width is to be decreased by 4 yards. It is desired that the new area be at least equal to the old. The new area is $(x + 5)(y - 4)$, and the old is xy. Thus we want $(x + 5)(y - 4) \geq xy$. Graph the solution to this inequality.

Graph the solution to the following systems of linear inequalities.

23. $2x - y > 5$, $x + y < 6$
24. $x < 3$, $y - x < 3$
25. $y < x + 4$, $x + 2y > -1$
26. $7x + 2y \geq 14$, $2x + 2y < 3$
27. $2x - 3 \leq y$, $3x - 6 \geq y$
28. $y - 2x > 4$, $2x + 5y \leq 10$
29. $x + 10y > -6$, $x - 4y > 3$
30. $4x + 13y < 52$, $\frac{2}{3}x + \frac{2}{5}y \geq -6$
31. $-y < 1$, $3x + 4y < 12$
32. $-6x - y > 0$, $-4x + 13y < 0$
33. $8x - 4y < -4$, $-3x + 3y > 3$
34. $2x - 3y < 11$, $10x + 2y > 13$
35. $\frac{3}{2}x + \frac{5}{4}y < 10$, $2x + 3y > -4$
36. $-4x - 3y > -5$, $6x + 12y > 3$
37. $10y < 6 - x$, $-4x - 2y < 10$
38. $x - 5y > 10$, $5x + y < 10$

39. The distance a vehicle with constant velocity travels is $d = rt$ (distance equals the product of rate and time). A car is traveling along a highway at a speed of 1.5 ± 0.5 kilometers per minute (kpm). Thus the rate r is at least 1 kpm, so the minimum distance traveled after t minutes is $d \geq 1t$. (a) Write a similar inequality that describes the maximum distance traveled, then (b) graph the solution to this system. Assume $t > 0$. The solution represents the set of possible distances traveled by the car after t minutes.

40. A box has dimensions that depend on two variables, x and y. Its length is $x + y - 3$ and its width is $2x - y + 6$. The length and width must have positive values. Thus, we require that $x + y - 3 > 0$ and $2x - y + 6 > 0$. Also, $x > 0$ and $y > 0$ must be true. Graph this system of inequalities.



In the following problems, maximize the value of the objective function P with regard to the constraints supplied. In all cases it is assumed that $x \ge 0$ and $y \ge 0$.

41. $$\begin{aligned} -x + y &\le 2 \\ x + y &\le 6 \\ P &= 2x + y \end{aligned}$$

42. $$\begin{aligned} -x + 2y &\le 4 \\ \tfrac{7}{6}x + y &\le 7 \\ P &= x + 2y \end{aligned}$$

43. $$\begin{aligned} x + 4y &\le 18 \\ 4x + 5y &\le 28 \\ P &= 2x + 3y \end{aligned}$$

44. $$\begin{aligned} -11x + 10y &\le 5 \\ -6x + y &\le -24 \\ P &= x + \tfrac{1}{2}y \end{aligned}$$

45. $$\begin{aligned} -26x + 21y &\le 14 \\ 2x + y &\le 12 \\ P &= 2x + \tfrac{1}{3}y \end{aligned}$$

46. $$\begin{aligned} 3x + 3y &\le 16 \\ 7x + 6y &\le 35 \\ P &= 2x + 3y \end{aligned}$$

47. $$\begin{aligned} 5x + 16y &\le 80 \\ -25x + 16y &\ge -100 \\ P &= 2x + 4y \end{aligned}$$

48. $$\begin{aligned} 2x + y &\le 10 \\ 4x + 3y &\le 24 \\ P &= x + y \end{aligned}$$

49. $$\begin{aligned} x + 4y &\le 20 \\ x + 2y &\le 12 \\ P &= \tfrac{1}{3}x + \tfrac{1}{2}y \end{aligned}$$

50. $$\begin{aligned} -x + y &\le 3 \\ 7x + 4y &\le 56 \\ P &= 2x - \tfrac{1}{2}y \end{aligned}$$

51. $$\begin{aligned} 2x + 5y &\le 25 \\ 3x + 2y &\le 21 \\ P &= x - \tfrac{1}{3}y \end{aligned}$$

52. $$\begin{aligned} x + 6y &\le 18 \\ 2x + y &\le 14 \\ P &= -x + 2y \end{aligned}$$

53. $$\begin{aligned} -3x + 4y &\le 20 \\ 2x + y &\le 16 \\ P &= -\tfrac{1}{2}x + 2y \end{aligned}$$

54. $$\begin{aligned} -3x + 8y &\le 28 \\ 2x + y &\le 13 \\ P &= x - y \end{aligned}$$

55. $$\begin{aligned} -x + 2y &\le 4 \\ x + 3y &\le 11 \\ x + y &\le 7 \\ P &= x + 2y \end{aligned}$$

56. $$\begin{aligned} -\tfrac{3}{2}x + y &\le 3 \\ x + y &\le 8 \\ 4x + y &\le 20 \\ P &= 2x + 3y \end{aligned}$$

57. $$\begin{aligned} -7x + 4y &\le 2 \\ y &\le 4 \\ \tfrac{4}{3}x + y &\le 12 \\ P &= 2x + y \end{aligned}$$

58. $$\begin{aligned} -2x + 5y &\le 15 \\ 2x + 3y &\le 17 \\ \tfrac{3}{2}x + y &\le 9 \\ P &= -x + 3y \end{aligned}$$

59. $$\begin{aligned} -x + y &\le 6 \\ 3x + 2y &\le 22 \\ x + y &\le 8 \\ P &= 3x - y \end{aligned}$$

60. $$\begin{aligned} \tfrac{1}{2}x + y &\le 8 \\ \tfrac{2}{3}x + y &\le \tfrac{26}{3} \\ 2x + y &\le 18 \\ P &= 12 - x - y \end{aligned}$$

61. $$\begin{aligned} y &\le 3 \\ \tfrac{1}{2}x + y &\le \tfrac{11}{2} \\ 4x + y &\le 30 \\ P &= \tfrac{1}{5}x - \tfrac{1}{8}y \end{aligned}$$

62. $$\begin{aligned} -4x + 3y &\le 3 \\ -x + 3y &\le 12 \\ 3x + y &\le 24 \\ P &= \tfrac{1}{2}x + 2y \end{aligned}$$

63. $$\begin{aligned} -2x + 3y &\le 15 \\ 5x + 3y &\le 36 \\ x + y &\le 8 \\ P &= 5x + 3y \end{aligned}$$

64. $$\begin{aligned} -x + y &\le 3 \\ 7x + 4y &\le 67 \\ x + y &\le 10 \\ P &= -x + \tfrac{1}{2}y \end{aligned}$$

65. $$\begin{aligned} -x + y &\le 8 \\ 5x + 6y &\le 70 \\ 5x + 2y &\le 40 \\ P &= 3x + y \end{aligned}$$

66. $$\begin{aligned} -8x + 9y &\le 27 \\ 10x + 7y &\le 94 \\ 2x + y &\le 18 \\ P &= 2x + 5y \end{aligned}$$

In the following problems, minimize the value of the objective function C with regard to the constraints supplied. In all cases it is assumed that $x \ge 0$ and $y \ge 0$.

67. $$\begin{aligned} -x + y &\ge 2 \\ x + y &\ge 6 \\ C &= 2x + y \end{aligned}$$

68. $$\begin{aligned} -x + 2y &\ge 4 \\ \tfrac{7}{6}x + y &\ge 7 \\ C &= x + 2y \end{aligned}$$

69. $$\begin{aligned} x + 4y &\ge 18 \\ 4x + 5y &\ge 28 \\ C &= 2x + 3y \end{aligned}$$

70. $$\begin{aligned} -11x + 10y &\ge 5 \\ -6x + y &\ge -24 \\ C &= x + \tfrac{1}{2}y \end{aligned}$$

71. $$\begin{aligned} -26x + 21y &\ge 14 \\ 2x + y &\ge 12 \\ C &= 2x + \tfrac{1}{3}y \end{aligned}$$

72. $$\begin{aligned} 3x + 3y &\ge 16 \\ 7x + 6y &\ge 35 \\ C &= 2x + 3y \end{aligned}$$

73. $$\begin{aligned} 4x + y &\ge 9 \\ x + y &\ge 6 \\ x + 5y &\ge 10 \\ C &= x + 2y \end{aligned}$$

74. $$\begin{aligned} x + y &\ge 9 \\ \tfrac{7}{2}x + y &\ge \tfrac{23}{2} \\ x + 3y &\ge 12 \\ C &= 2x + 3y \end{aligned}$$

75. $$\begin{aligned} \tfrac{7}{2}x + y &\ge 8 \\ x + 7y &\ge 9 \\ x + y &\ge 4 \\ C &= 2x + y \end{aligned}$$

76. $$\begin{aligned} x + y &\ge 8 \\ 2x + 3y &\ge 17 \\ \tfrac{3}{2}x + y &\ge 9 \\ C &= 5x + 4y \end{aligned}$$

77. $$\begin{aligned} -x + y &\ge -2 \\ 3x + 2y &\ge 22 \\ x + y &\ge 8 \\ C &= 3x - y \end{aligned}$$

78. $$\begin{aligned} 5x + y &\ge 6 \\ x + 5y &\ge 6 \\ \tfrac{3}{4}x + y &\ge 3 \\ C &= 12 - x - y \end{aligned}$$

79. A furniture company produces two products, tables and chairs. Tables sell for \$29 while chairs sell for \$10. It takes 3 hours to assemble a table and 1 hour to assemble a chair. In a production run there are 300 hours available for assembly. It takes 2 hours to finish a table, and $\frac{5}{6}$ of an hour for a chair. The finishing department has 240 hours available in a production run. Maximize total income from the sales of tables and chairs.

80. In a company there are assembly and paint departments. The company produces two products, A and B. Product A requires 2 hours for assembly and 1 hour for painting; B requires 5 hours for assembly and 2 hours for painting. A sells for \$3, and B for \$10. The assembly department is limited to 200 hours for a production run, and the paint department to 100 hours. How many of each product should be produced to maximize the income from sales?

81. A coal mining company has crews comprised of workers and excavation machines. It has two types of crews, type A and type B. Type A crews have 12 workers and 2 machines, and type B crews have 20 workers and 4 machines. Type A crews produce 13 tons of coal per hour and type B crews produce 25 tons per hour. The company has 260 workers and 50 machines. How many crews of each type should be allocated to maximize coal production?

82. A large logging company has two types of crews of workers and supervisors, called A-crews and B-crews. It has found that A-crews, which have a crew of 1 supervisor and 4 loggers, log 20 trees per day, and that B-crews, which have a crew of 2 supervisors and 6 loggers, log 30 trees per day. The company has 40 supervisors and 150 loggers on its payroll. What mix of crews would produce the most trees per day?

$\boxed{83.}$ A certain animal food is available from two sources, A and B. A supplies 5 grams per pound (5 gm/lb) of protein and 4 gm/lb of carbohydrates. B supplies 8 gm/lb and 3 gm/lb of protein and carbohydrates. A costs 15 cents per pound, and B is 18 cents. If the *minimum* daily requirement for this animal is 20 gm of protein and 12 gm of carbohydrates, how much of A and B should be used to minimize costs?

84. A company builds toy automobiles in two sizes, large and small. The large size sells for \$3, and the small for \$1. The following constraints apply to producing these products. The large car requires 14 ounces of plastic to produce, and the small car requires 6 ounces of plastic to produce. The company is limited to 6,000 ounces of plastic. It takes 2 minutes to produce a small car, and 3.5 minutes for a large. Total labor is limited to 2,000 minutes. The small cars require 5 small decorative decals, while the large cars only require one of the same type. The company has 2,000 of these small decals. How many of each size should be built to maximize the total dollar amount of sales?

Skill and review

1. Write the identity matrix of order 4.
2. Rewrite $\left|3 - \frac{\sqrt{2}}{2}\right|$ without absolute value notation.
3. Solve $|4 - x| < 10$.
4. Solve $2x - 3 = 0$.
5. Solve $2x^2 - 3x = 5$.
6. Solve $|2x^2 - 3x| = 5$.
7. Simplify $\sqrt[3]{16x^5y^2z}$.
8. Graph $f(x) = |x - 2|$.

10–5 Systems of linear equations—matrix algebra

An insect lives in three stages: egg, larval, and adult. In the first stage females have no progeny. In the second they have four daughters. In the third they have three daughters. The survival rate in stage 1 is 22.9%, and in stage 2 it is 12.5%. Assume the number of females in each stage in some initial generation is 1,000. Find the number of females in each stage after three generations.

This is one of many, many types of problems that can be solved using matrix algebra, the topic of this section.

In this section we will also see that systems of linear equations can be described in terms of matrices. This process has developed into an area of mathematics called linear algebra, which finds wide application in the social sciences as well as engineering and mathematics.

Addition and subtraction of matrices

Operations of addition, subtraction, and multiplication are defined for matrices. Addition and subtraction are defined in a natural way. To add or subtract matrices we add or subtract each element.

> **Addition and subtraction of matrices**
> Let A and B be two matrices of the same dimensions, $m \times n$. $C = A + B$ is an $m \times n$ matrix in which $c_{i,j} = a_{i,j} + b_{i,j}$, and $D = A - B$ is an $m \times n$ matrix in which $d_{i,j} = a_{i,j} - b_{i,j}$ for $1 \le i \le m$, $1 \le j \le n$.

Observe that these operations are not defined for matrices whose dimensions differ.

The TI-81, as well as most graphing calculators, can store matrices and do most matrix operations. This will be illustrated in example 10–5 F.

Example 10–5 A illustrates addition and subtraction of matrices.

■ Example 10–5 A

Perform the indicated addition or subtraction.

1. $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} (2+3) & (-1-4) \\ (3-3) & (5+0) \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 0 & 5 \end{bmatrix}$

2. $\begin{bmatrix} -4 & \frac{1}{2} \\ 2 & -3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & \frac{3}{2} \\ 2 & 5 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} (-4+3) & (\frac{1}{2}-\frac{3}{2}) \\ (2-2) & (-3-5) \\ (0+6) & (1-1) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -8 \\ 6 & 0 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 2 & 1 \\ 5 & -2 & 0 \end{bmatrix}$ Not defined since the dimensions are not the same. ■

The scalar product

There are two forms of multiplication of matrices. The first is **scalar multiplication.** In the context of matrix algebra, real numbers are also called **scalars.** The product of a scalar and a matrix is a matrix in which each element is the product of the scalar and the element.

> **Scalar product**
> If A is a matrix of dimension $m \times n$ and $k \in R$, then the scalar product of k and A is $C = kA$, where $c_{i,j} = ka_{i,j}$ for $1 \le i \le m$, $1 \le j \le n$.

■ *Example 10–5 B*

Form the scalar product of -5 and $A = \begin{bmatrix} -6 & 5 & 1 \\ 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix}$.

$$-5A = -5\begin{bmatrix} -6 & 5 & 1 \\ 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -5(-6) & -5(5) & -5(1) \\ -5(2) & -5(-1) & -5(3) \\ -5(0) & -5(4) & -5(-2) \end{bmatrix}$$
$$= \begin{bmatrix} 30 & -25 & -5 \\ -10 & 5 & -15 \\ 0 & -20 & 10 \end{bmatrix}$$

■

The dot product

Before we talk about the second form of matrix multiplication we discuss another "product," the dot product.

An **n-dimension vector** is a matrix of n values, in which the number of rows or columns is 1. Thus, $[2, -4]$ is a 2-dimensional vector, $[-3, 4, 0, 11]$ is a 4-dimensional vector, and $\begin{bmatrix} 9 \\ 3 \\ -5 \end{bmatrix}$ is a 3-dimensional vector. Since one of the dimensions is one we often simplify our notation for a matrix element. In this case, if V is an n-dimensional vector then v_i, $1 \leq i \leq n$ describes the n elements in V.

The **dot product** of two n-dimensional vectors is a scalar (real number) that is the sum of the products of the corresponding elements in the vectors. The dot product is indicated by the symbol $\cdot$.

Dot product of two n-dimensional vectors

Let U and V be two n-dimensional vectors. Then the dot product k of U and V, $k = U \cdot V$, is a scalar such that

$$k = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Observe that the dot product requires that the vectors have the same number of elements. Example 10–5 C illustrates forming the dot product.

■ *Example 10–5 C*

Form the dot product of the given vectors.

1. $[-2, 0, 3, 1] \cdot [4, -5, 10, 6] = (-2)(4) + (0)(-5) + (3)(10) + (1)(6)$
$= 28$

2. $[3, 1, -2] \cdot \begin{bmatrix} 4 \\ -4 \\ 5 \end{bmatrix} = (3)(4) + (1)(-4) + (-2)(5) = -2$

■

The matrix product

The **matrix product** is defined under certain conditions placed on the dimensions.

> **Matrix product**
> If A is an $m \times k$ matrix, and B is a $k \times n$ matrix, then the matrix product $C = AB$ is an $m \times n$ matrix where $c_{i,j}$ is the dot product of the ith row of A and the jth column of B.

For the matrix product to be defined the number of columns of the first factor must equal the number of rows of the second factor. The following illustrates this, as well as how the dimensions of the result are determined.

$$\begin{array}{ccccc} A & & B & = & C \\ m \times k & & k \times n & & m \times n \end{array}$$

$$k = k$$

dimensions of result

For an example we will form the product of $A = \begin{bmatrix} -2 & 0 \\ 5 & 1 \\ 2 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & 2 & 5 \\ 1 & 3 & -1 & 3 \end{bmatrix}$. A is a 3×2 matrix and B is a 2×4 matrix, so $C = AB$ will be a 3×4 matrix.

The element in the second row, third column of C, $c_{2,3}$, is the dot product of the vectors that are the second row of A and the third column of B; thus,

$$c_{2,3} = [5\ 1] \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 5(2) + 1(-1) = 9$$

$$\begin{bmatrix} -2 & 0 \\ 5 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 5 \\ 1 & 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 2 & -4 & -10 \\ 21 & -2 & 9 & 28 \\ 4 & -14 & 8 & -2 \end{bmatrix}$$

As another example:

$$c_{1,4} = [-2,0] \cdot \begin{bmatrix} 5 \\ 3 \end{bmatrix} = -2(5) + 0(3) = -10$$

$$\begin{bmatrix} -2 & 0 \\ 5 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 5 \\ 1 & 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 2 & -4 & -10 \\ 21 & -2 & 9 & 28 \\ 4 & -14 & 8 & -2 \end{bmatrix}$$

The complete array is

$$\begin{bmatrix} -2 & 0 \\ 5 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 5 \\ 1 & 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 2 & -4 & -10 \\ 21 & -2 & 9 & 28 \\ 4 & -14 & 8 & -2 \end{bmatrix}$$

Example 10–5 D also illustrates matrix multiplication.

■ ***Example 10–5 D*** Compute each product.

1. $\begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & -2 & 0 \\ 1 & 3 & -4 \end{bmatrix}$

This is the product of a 2×2 and a 2×3 matrix; the result will be a 2×3 matrix.

$$= \begin{bmatrix} (2)(4) + (-3)(1) & (2)(-2) + (-3)(3) & (2)(0) + (-3)(-4) \\ (-1)(4) + (5)(1) & (-1)(-2) + (5)(3) & (-1)(0) + (5)(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -13 & 12 \\ 1 & 17 & -20 \end{bmatrix}$$

2. $\begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}[-10\ 15]$

$(3 \times 1) \times (1 \times 2)$ tells us that the multiplication is defined, and that the result will be a 3×2 matrix.

$$\begin{bmatrix} (6)(-10) & (6)(15) \\ (2)(-10) & (2)(15) \\ (-2)(-10) & (-2)(15) \end{bmatrix} = \begin{bmatrix} -60 & 90 \\ -20 & 30 \\ 20 & -30 \end{bmatrix}$$

3. $\begin{bmatrix} 5 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a + c & 5b + d \\ 2a - 3c & 2b - 3d \end{bmatrix}$

4. $\begin{bmatrix} -2 & 1 \\ 0 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \\ 4 & 4 \end{bmatrix}$; this product is not defined.

$$(3 \times \underbrace{2) \times (3}_{\text{not equal}} \times 2)$$

■

After developing several necessary facts we will show how to solve systems of equations by matrix multiplication.

The identity matrix

We defined the order-n identity matrix in section 10–2. It is a square matrix of n rows and n columns, with every element zero except on the main diagonal (upper-left to lower-right), where every element is a 1. Whenever an element is on the main diagonal the row number is the same as the column number. This is used for a more formal definition.

Identity matrix of order-*n*

The identity matrix of order n is the $n \times n$ matrix I_n such that

$$i_{a,b} = \begin{cases} 1 \text{ if } a = b \\ 0 \text{ if } a \neq b \end{cases} \quad 1 \leq a, b \leq n.$$

Note Where the value of n is not important or implied we often use I instead of I_n.

The matrix I_n is called the **identity element for matrix multiplication,** because it can be shown that if A is a square matrix then $IA = AI = A$. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix}$$ Multiplication by I_2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$$ Multiplication by I_3

Inverse of a matrix

For a *square* matrix C, if there is a matrix M such that $CM = MC = I$, we say that M is the inverse of C; we then usually call this matrix C^{-1} (the same notation we use for the inverse of a function). As with functions the superscript "-1" is not an exponent. It simply means "the matrix multiplicative inverse of."

Multiplicative inverse of a square matrix

For a square matrix C, if there exists a matrix C^{-1} such that

$$C^{-1}C = I \text{ and } CC^{-1} = I$$

we call C^{-1} the multiplicative inverse of C.

It can be proven that a square matrix C has a multiplicative inverse if and only if its determinant $|C|$ is nonzero. One method of finding the multiplicative inverse of a square matrix is shown below. In this procedure, we form the appropriate **augmented matrix.** This means to create a new matrix of twice as many columns as the original, where the rightmost columns form the identity matrix.

■ ***Example 10–5 E***

Find the multiplicative inverse of the given matrix.

1. $A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

$$\left[\begin{array}{cccc} 2 & -5 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right]$$ Form the augmented matrix

Now transform the left two columns into I_2 using row operations (see section 10–2).

$$\left[\begin{array}{cccc} 2 & -5 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right]$$ Use -1 to sweep out column 2

$$\left[\begin{array}{cccc} -13 & 0 & 1 & -5 \\ 3 & -1 & 0 & 1 \end{array}\right]$$ $R1 \leftarrow -5(R2) + R1$

This notation means that row 1 is replaced with the sum of -5 times row 2 and row 1. We will now use -13 to sweep out column 1.

$$\left[\begin{array}{cccc} -13 & 0 & 1 & -5 \\ 0 & -13 & 3 & -2 \end{array}\right]$$ $R2 \leftarrow 3(R1) + 13(R2)$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{13} & \frac{5}{13} \\ 0 & 1 & -\frac{3}{13} & \frac{2}{13} \end{bmatrix}$$ Divide each row by -13

Observe that we have I_2 in the left two columns.

$$A^{-1} = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ -\frac{3}{13} & \frac{2}{13} \end{bmatrix}$$

This can be verified by computing $A^{-1}A$ and AA^{-1}, and noting that each product produces I_2.

2. $C = \begin{bmatrix} 1 & -3 & 0 \\ 2 & 0 & -1 \\ -2 & 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ -2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$ Form the augmented matrix

Now transform the left three columns into I_3. That is sweep out columns 1, 2, and 3. First sweep out column 3, using row 2.

$$\begin{bmatrix} 1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 6 & 3 & 0 & 0 & 4 & 1 \end{bmatrix}$$ R3 ← 4R2 + R3

Column 3 is now swept out, using row 2. We can now only use rows 1 or 3 to sweep out columns 1 and 2. We use the -3 in column 2.

$$\begin{bmatrix} 1 & -3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 & 4 & 1 \end{bmatrix}$$ R3 ← R3 + R1

Now columns 2 and 3 are swept out, and only row 3 may be used as a key row.

$$\begin{bmatrix} 0 & -21 & 0 & 6 & -4 & -1 \\ 0 & 0 & -7 & -2 & -1 & -2 \\ 7 & 0 & 0 & 1 & 4 & 1 \end{bmatrix}$$ R1 ← 7(R1) − R3
R2 ← 7(R2) − 2(R3)

$$\begin{bmatrix} 7 & 0 & 0 & 1 & 4 & 1 \\ 0 & -21 & 0 & 6 & -4 & -1 \\ 0 & 0 & -7 & -2 & -1 & -2 \end{bmatrix}$$ Rearrange the order of the rows

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{7} & \frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{4}{21} & \frac{1}{21} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$ Divide each row by its first nonzero element

$$C^{-1} = \begin{bmatrix} \frac{1}{7} & \frac{4}{7} & \frac{1}{7} \\ -\frac{2}{7} & \frac{4}{21} & \frac{1}{21} \\ \frac{2}{7} & \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

This can be verified by checking that $C^{-1}C = CC^{-1} = I_3$. ■

Example 10–5 F illustrates how to perform matrix operations on the TI-81 graphing calculator. This calculator can store up to three matrices of size up to 6×6.

■ Example 10–5 F

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ -3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0.5 & -4 \\ 0 & 1 & 6 \\ -0.5 & 1 & 4 \end{bmatrix}$. Use the TI-81 to solve each problem.

1. Enter each matrix into the TI-81.

$\boxed{\text{MATRX}}$ EDIT 1 3 $\boxed{\text{ENTER}}$ 3 $\boxed{\text{ENTER}}$ 2 $\boxed{\text{ENTER}}$ $\boxed{(-)}$ 1 $\boxed{\text{ENTER}}$ 3 $\boxed{\text{ENTER}}$ 0 $\boxed{\text{ENTER}}$ 4 $\boxed{\text{ENTER}}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$ $\boxed{(-)}$ 3 $\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$ 5 $\boxed{\text{QUIT}}$

$\boxed{\text{[A]}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{[A]}}$ is $\boxed{\text{2nd}}$ 1; The array A appears on the display.

To enter B, start with $\boxed{\text{MATRX}}$ EDIT 2 3 $\boxed{\text{ENTER}}$ 3 $\boxed{\text{ENTER}}$, then enter the array's values as for A.

2. Compute $3A + B$.

3 $\boxed{\text{[A]}}$ $\boxed{+}$ $\boxed{\text{[B]}}$ $\boxed{\text{ENTER}}$

$$\begin{bmatrix} 8 & -2.5 & 5 \\ 0 & 13 & 0 \\ -9.5 & 4 & 19 \end{bmatrix}$$

3. Compute AB.

$\boxed{\text{[A]}}$ $\boxed{\text{[B]}}$ $\boxed{\text{ENTER}}$

$$\begin{bmatrix} 2.5 & 3 & -2 \\ 1 & 2 & 16 \\ -8.5 & 4.5 & 38 \end{bmatrix}$$

4. Compute $|A|$.

$\boxed{\text{MATRIX}}$ 5 $\boxed{\text{[A]}}$ $\boxed{\text{ENTER}}$ $|A| = 74$

5. Compute A^{-1}; round each element to four decimal places.

$\boxed{\text{MATH}}$ NUM 1 $\boxed{\text{[A]}}$ $\boxed{x^{-1}}$ $\boxed{,}$ 4 $\boxed{)}$ $\boxed{\text{ENTER}}$ $\boxed{,}$ is $\boxed{\text{ALPHA}}$ $\boxed{.}$

Use the round function to show four decimal places. Use the left and right arrow keys to see the full array.

$$\begin{bmatrix} 0.2973 & 0.1081 & -0.1351 \\ 0.0811 & 0.2568 & 0.0541 \\ 0.1622 & 0.0135 & 0.1081 \end{bmatrix}$$

To show exact values (rational numbers) compute $|A|A^{-1}$. This gives the numerators of the result's elements. The denominators are $|A|$. On the TI-81 you must use the $\boxed{\times}$ (multiply) key between $|A|$ and A^{-1}. ■

Solving systems of equations by matrix multiplication

We define **equality of matrices** to mean that *two matrices are equal if and only if each element of one matrix is equal to each element of the other.* This implies that two matrices can be equal only if their dimensions are identical. By way of example, if we state that $\begin{bmatrix} 5 & b \\ c & 3 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ then we know that $x = 5$, $y = b$, $z = c$, and $w = 3$.

The **multiplication property of equality** can be proved for matrices A, X, and Y:

$$\text{if } X = Y, \text{ then } AX = AY.$$

Associativity states that when multiplying the expression ABC we can multiply AB first or BC first. Specifically, if A, B, and C are matrices then

$$A(BC) = (AB)C$$

This property can also be proved true for matrix multiplication in which all the products are defined (i.e., the dimensions match up properly).

A system of n linear equations in n variables can be described using matrix multiplication and the definition of equality of matrices. For example,

$$\begin{aligned} 3x - 2y &= 5 \\ x + 3y &= -9 \end{aligned}$$

can be described as

$$\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

because we can recreate the equations as follows.

$$\begin{bmatrix} 3x - 2y \\ x + 3y \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$ Multiply the matrices in the left member

$$\begin{aligned} 3x - 2y &= 5 \\ x + 3y &= -9 \end{aligned}$$ By equality of matrices

If we let C represent the coefficient matrix $\begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix}$, X represents the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, and K represent the matrix of constants $\begin{bmatrix} 5 \\ -9 \end{bmatrix}$, then the system above can be described by the matrix equation

$$CX = K$$

If a matrix C^{-1} could be found such that $C^{-1}C = I$, we could solve the system by the following process.

$CX = K$	The original system of equations expressed using matrices
$C^{-1}CX = C^{-1}K$	Multiply by C^{-1}
$IX = C^{-1}K$	$C^{-1}C = I$
$X = C^{-1}K$	X is the matrix of variables

Example 10–5 G illustrates how to solve a system of linear equations using matrix multiplication.

■ ***Example 10–5 G***

Solve the system of equations using matrix multiplication.

1. $2x - 5y = 3$
$3x - y = -4$

This system can be described by the matrix equation $CX = K$ as shown.

$$\underset{C}{\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{K}{\begin{bmatrix} 3 \\ -4 \end{bmatrix}}$$

Thus, as computed above, we need to compute $C^{-1}K$. We found C^{-1} in example 10–5 E, so we can proceed:

$X = C^{-1}K$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ -\frac{3}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ — Replace C^{-1} and K

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{23}{13} \\ -\frac{17}{13} \end{bmatrix}$ — Multiply the right member

$x = -\frac{23}{13}$ and $y = -\frac{17}{13}$ — Use equality of matrices

2. $x - y = 3$
$2y + 3z = -2$
$-2x + 3y = 5$

We write the system as a matrix product, then solve the matrix equation for the variable array X.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$
$$CX = K$$

so

$$X = C^{-1}K$$

Thus we must find C^{-1} to find X.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Augmented matrix

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

Row operations used to form I_3 in the first three columns

$$C^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ -\frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Thus we proceed,

$$\begin{array}{ccccc} X & = & C^{-1} & & K \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ -\frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} & & \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ -8 \end{bmatrix} \end{array}$$

$x = 14$, $y = 11$, $z = -8$

To solve this problem on the TI-81 we enter the array C and K. Since this calculator calls its arrays A, B, and C, we use the terminology

$$X = A^{-1}B$$

First, matrices A and B (that is, C and K above) are entered into the TI-81 as shown in example 10–5 F.

Now compute the matrix product $A^{-1}B$:

$\boxed{[A]}$ $\boxed{x^{-1}}$ $\boxed{[B]}$ $\boxed{\text{ENTER}}$

The result appears: $\begin{bmatrix} 14 \\ 11 \\ -8 \end{bmatrix}$. Thus $(x,y,z) = (14,11,-8)$.

Mastery points

Can you

- Form the dot product of vectors?
- Compute scalar and matrix products?
- Find the inverse of a matrix by using an augmented matrix?
- Use matrix multiplication and the inverse of a matrix to solve systems of linear equations?

Exercise 10–5

Add or subtract the given matrices as indicated.

1. $\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & -5 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -6 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix}$

5. $\begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & -3 \end{bmatrix}$

6. $\begin{bmatrix} -4 & 1 & 3 \\ -6 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 5 \\ 0 & -6 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 3 & -1 \\ 2 & 15 & -2 \end{bmatrix}$

8. $\begin{bmatrix} -5 & 2 & 6 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 13 & -12 & 5 \\ 10 & 1 & 0 \end{bmatrix}$

9. $\begin{bmatrix} -1 & -2 \\ -1 & 3 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -2 & 1 \\ -2 & -3 \end{bmatrix}$

10. $\begin{bmatrix} -4 & 1 \\ 3 & -6 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ 5 & 0 \\ -6 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 \\ 3 & -2 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -1 & 2 \\ 15 & -2 \end{bmatrix}$

12. $\begin{bmatrix} -5 & 2 \\ 6 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 13 & -12 \\ 5 & 10 \\ 1 & 0 \end{bmatrix}$

Compute the given scalar product.

13. $4\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

14. $3\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

15. $-5\begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$

16. $-2\begin{bmatrix} 8 & -6 \\ 1 & 10 \end{bmatrix}$

17. $\frac{1}{2}\begin{bmatrix} 4 & 1 & -2 \\ 2 & 6 & -2 \end{bmatrix}$

18. $\frac{2}{3}\begin{bmatrix} -15 & 9 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

19. $-1\begin{bmatrix} 2 & -13 \\ 5 & 0 \\ 3 & 2 \end{bmatrix}$

20. $0\begin{bmatrix} 0 & 2 \\ -3 & 2 \\ -3 & 5 \end{bmatrix}$

Form the dot product of the given vectors.

21. $[3, -4], [-2, 5]$

22. $[11, 0, -3, 2], [4, -2, 2, 6]$

23. $[3, 1, -2], \begin{bmatrix} 4 \\ -4 \\ 5 \end{bmatrix}$

24. $[-2, 5], [3, 1]$

25. $\left[\sqrt{2}, \frac{1}{3}, -5\right], \left[\sqrt{8}, 6, \frac{\pi}{5}\right]$

26. $[-4, 0, 3, -5], \begin{bmatrix} -4 \\ 2 \\ 0 \\ 2 \end{bmatrix}$

27. Find a vector v such that $[-3, 1, -2, 5] \cdot v = 1$.

28. Find a vector v such that $[5, 2, -4, 3] \cdot v = \frac{1}{2}$

Compute the indicated matrix products.

29. $\begin{bmatrix} -1 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & -5 \end{bmatrix}$

30. $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix}$

31. $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

32. $\begin{bmatrix} 3 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ -2 & 5 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 & -1 \\ 5 & 2 & 0 \\ 1 & -2 & -3 \end{bmatrix}$

34. $\begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & -6 & 2 \\ 2 & 0 & -1 \end{bmatrix}$

35. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -3 & 5 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & 2 \\ 15 & -2 \end{bmatrix}$

36. $\begin{bmatrix} -5 & 2 & 6 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & -12 \\ 5 & 10 & 1 \\ 3 & 4 & 0 \end{bmatrix}$

37. $\begin{bmatrix} -1 & -2 & -1 \\ 3 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & 1 \\ -2 & -3 \end{bmatrix}$

38. $\begin{bmatrix} -4 & 1 & 3 \\ -6 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 5 & 0 \\ -6 & 2 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & 2 \\ 15 & -2 \end{bmatrix}$

40. $\begin{bmatrix} -5 & 2 & 6 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 13 & -12 \\ 5 & 10 \\ 1 & 0 \end{bmatrix}$

41. $\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -4 & -21 & 10 \\ 1 & 3 & -4 \end{bmatrix}$

42. $[-10, 1, -5]\begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$

43. $\begin{bmatrix} 5x & 1 \\ 4y & -3 \end{bmatrix}\begin{bmatrix} -4x & 3 \\ y & 9 \end{bmatrix}$

44. $\begin{bmatrix} x & 2 \\ 4y+1 & 2 \end{bmatrix}\begin{bmatrix} -4x-1 & 3 \\ y+1 & 2 \end{bmatrix}$

45. $\begin{bmatrix} 2 & 3 & -1 \\ 4 & \frac{1}{2} & 8 \end{bmatrix}\begin{bmatrix} -4 & -6 \\ 10 & 2 \\ 3 & -4 \end{bmatrix}$

46. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} -1 & -6 & 1 & 3 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

47. $\begin{bmatrix} 2 & 3 & 2 & -1 \\ 4 & -6 & 2 & 8 \end{bmatrix}\begin{bmatrix} -4 & -6 & 5 \\ 11 & 1 & 3 \\ -4 & -2 & -7 \\ 2 & 0 & -3 \end{bmatrix}$

48. $\begin{bmatrix} 1 & -6 & 2 & 3 \\ 2 & -1 & 4 & 8 \end{bmatrix}\begin{bmatrix} -4 & -6 & 3 \\ -4 & -2 & -7 \\ 2 & 5 & 11 \\ 1 & 0 & -3 \end{bmatrix}$

49. Let $A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 & 2 & 7 \\ -1 & 3 & -6 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 2 & 0 \\ 2 & 5 & -1 \\ 8 & 3 & -4 \end{bmatrix}$. By computation determine whether $(AB)C = A(BC)$.

50. In the text it was stated that a square matrix has an inverse if and only if its determinant is not zero. The determinant of matrix $A = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 1 & -3 \\ -1 & -1 & 11 \end{bmatrix}$ is zero. (a) Verify this by computation. (b) Attempt to find A^{-1} and observe what happens.

Find the inverse of each matrix. If the matrix does not have an inverse, state this. Also see problem 50.

51. $\begin{bmatrix} 12 & -5 \\ -3 & -1 \end{bmatrix}$

52. $\begin{bmatrix} 5 & -5 \\ -3 & 4 \end{bmatrix}$

53. $\begin{bmatrix} -3 & \frac{1}{2} \\ 2 & 3 \end{bmatrix}$

54. $\begin{bmatrix} 1 & -\frac{2}{3} \\ 6 & -1 \end{bmatrix}$

55. $\begin{bmatrix} 0 & -3 & 0 \\ 2 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}$

56. $\begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & -1 \\ 3 & 0 & 4 \end{bmatrix}$

57. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

58. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}$

59. $\begin{bmatrix} 0 & -3 & 0 & 2 \\ 2 & -1 & 0 & 3 \\ -5 & 2 & 0 & -1 \\ 2 & 0 & -6 & 1 \end{bmatrix}$

60. $\begin{bmatrix} 2 & -3 & 0 & 0 \\ 2 & -2 & 1 & 3 \\ -3 & 2 & 0 & -1 \\ 2 & 0 & -2 & 1 \end{bmatrix}$

61. $\begin{bmatrix} 0 & 0 & 4 & 1 \\ 2 & -1 & 0 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 2 & 6 & 1 \end{bmatrix}$

62. $\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & -1 & 0 & 3 \\ 5 & 1 & -2 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$

Solve the system of equations using matrix multiplication. *The matrix of coefficients of each problem corresponds to the matrix from the problem indicated in parentheses, where its inverse was computed.* Thus, there is no need to recompute the inverse of this matrix.

63. $12x - 5y = -6$ (51)
$-3x - y = -3$

64. $5x - 5y = -15$ (52)
$-3x + 4y = 10$

65. $-3x + \frac{1}{2}y = -7$ (53)
$2x + 3y = -2$

66. $x - \frac{2}{3}y = -5$ (54)
$6x - y = -12$

67. $-3y = 3$ (55)
$2x - z = 1$
$2x + 3y + 4z = 13$

68. $x + 3y + 2z = 3$ (56)
$-2x - z = 2$
$3x + 4z = 2$

69. $x + 3z = -6$ (57)
$2x - z = -5$
$2x + 3y = -4$

70. $2x + 3y + z = -4$ (58)
$y + z = -1$
$2x + 4z = 5$

71. $-3y + 2w = 8$ (59)
$2x - y + 3w = 7$
$-5x + 2y - w = -10$
$2x - 6z + w = -9$

72. $2x - 3y = -8$ (60)
$2x - 2y + z + 3w = 2$
$-3x + 2y - w = 6$
$2x - 2z + w = -11$

73. $4z + w = -2$ (61)
$2x - y + 3w = 5$
$3x + 2y + w = 11$
$2x + 2y + 6z + w = 4$

74. $x + 3z - 2w = 21$ (62)
$-y + 3w = -7$
$5x + y - 2z - w = 3$
$2y + z + w = 5$

Solve the system of equations using matrix multiplication.

75. $x + \frac{2}{3}y = -1$
$-3x + y = 12$

76. $2x + \frac{1}{2}y = -1$
$3x - y = -5$

77. $-2x + y = 1$
$4x - y = 0$

78. $-x + 3y = 13$
$2x + y = 2$

79. $x + y - 5z = -9$
$-x + 2z = 6$
$2x + 2y = 2$

80. $-y + 3z = 9$
$-2x + 2y + z = 1$
$x + 3y + 2z = 7$

81. $-x - y - 2z = 1$
$x + y - 4z = -4$
$-\frac{1}{2}y + 2z = 3$

82. $-2x + 2z = -8$
$4x - y - z = 8$
$x + 2y - 2z = 5$

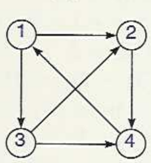
In the following problems use the three scalars $a = 2$, $b = -3$, and $c = 5$, and two matrices, $A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 4 & -1 \end{bmatrix}$, to compute the values of the following polynomial matrix expressions.

83. aAB **84.** bA^2 **85.** $aA - bB$ **86.** $bA^2 - aB^2 - cB$ **87.** $(A - B)^2$ **88.** $AB - BA$

Solve the following problems.

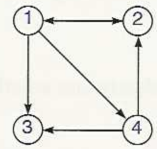
89. In archeology the problem of placing sites and artifacts in proper chronological order is called sequence dating.[6] In one situation with five types of pottery and four graves, the computation
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
must be made. The result can be used to put the grave sites in chronological order. Perform this computation.

90. The figure is from graph theory, an area of mathematics that has application in the sciences and business. It shows four nodes labeled 1 through 4 and paths from some nodes to others. This graph can be represented by the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, where a 1 in row i column j means there is a path from node i to node j. For example, there is a 1 in row 2 column 4, which shows that node 2 has a path to node 4. The rest of the row is zeros because node 2 has a path only to node 4. If we compute $A^2 = A \cdot A$ the matrix would give the number of paths of length 2 from one node to the other. Perform this computation and use the result to determine how many paths of length 2 there are from node 1 to node 4.



[6]From "Mathematics in Archaeology," by Gareth Williams, Stetson University, from *The Two-Year College Mathematics Journal,* Vol. 13, No. 1, January 1982.

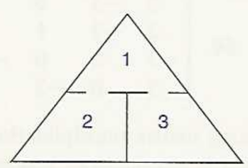
91. Create a matrix A that describes the graph in the figure (see problem 90). Compute A^2 and determine how many paths of length 2 there are from node 1 to node 2.



92. Referring to problem 90, compute A^3 and determine how many paths of length 3 there are from node 1 to node 4.

93. Referring to problem 91 compute A^3 and determine how many paths of length 3 there are from node 1 to node 2.

94. The figure could represent many situations in which there are three states. We will imagine it is a maze with three rooms. If a mouse is put in room 1, it has two choices for moving into another room. Assuming it

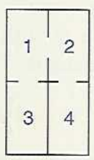


chooses randomly the chance of moving into either room 2 or 3 is $\frac{1}{2}$. In rooms 2 or 3, it has no choices for moving into another room. Thus the chance of moving into room 1 from room 2 is 1. The array $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ can describe this situation, where the entry in row i column j describes the probability that, if the mouse is in room i it will next move to room j. In the array $A^2 = A \cdot A$, $a_{i,j}$ would tell the probability that, if the mouse started in room i, it is in room j after two room changes. Compute A^2 and determine the probability that a mouse which

started in room 2 is in room 3 after 2 room changes (i.e., that it went from room 2 to room 1 to room 3).

The basic idea presented here is used to create what is called Markov chains (the powers of the matrix A) to study manufacturing and biological processes, economies, and chemical reactions among other things.

$\boxed{95.}$ The array for the maze in the figure is $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (see problem 94). Compute A^2 and use it to determine the probability that a mouse which started in room 2 is in room 3 after two moves.



96. Referring to problem 95, compute A^3 and determine the probability that a mouse that started in room 3 is in room 4 after 3 moves.

97. Let $L = \begin{bmatrix} 0 & 4 & 3 \\ 0.229 & 0 & 0 \\ 0 & 0.125 & 0 \end{bmatrix}$. This represents a creature that lives in three stages. An example would be an insect that has an egg, larval, and adult stage. In the first stage, females have no progeny. In the second, they have four daughters. In the third, they have three daughters. The survival rate in stage one is 22.9%, and in stage two it is 12.5%. Let $V = \begin{bmatrix} 1{,}000 \\ 1{,}000 \\ 1{,}000 \end{bmatrix}$ represent the number of females in each stage in some initial generation. (L is called a LESLIE matrix.) $LV = \begin{bmatrix} 7{,}000 \\ 229 \\ 125 \end{bmatrix}$ is the number of females in each stage after one life cycle (generation), and in general L^nV is the number after n cycles. (The elements of each state are rounded to the nearest integer.)

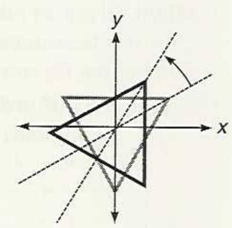
Compute L^3V, and find the number of females in each stage after three cycles (generations).

98. If a figure, such as the triangle in the figure, is to be rotated through an angle θ (theta), every point (x,y) must be transformed to a new point (x',y'). It can be shown that the new point can be found by matrix multiplication of the original point. The matrix is $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Note that $\cos\theta$ (read cosine of angle theta) and $\sin\theta$ (read sine of angle theta) are functions defined in trigonometry. They give a certain value for any angle θ.

If $P = (x,y)$ is to be rotated, the new point $P' = P \cdot R$. (Note that R is on the *right* of the point P.) For an angle of rotation of 30°, $R = \begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. Assume a triangle is defined by the points $(3,2)$, $(-3,2)$, and $(0,-4)$.

a. Draw this triangle.

b. Rotate each point by 30° by using the matrix R as given above, then draw the rotated triangle determined by these three new points.



Skill and review

1. Graph the system of inequalities $\begin{matrix} 2x + y < 6 \\ y \le 3x - 1 \end{matrix}$.

2. Solve the system of equations

$2x - y - 2z = -7$

$x + y + 4z = 2$

$3x + 2y - 2z = -3$

3. Find the point at which the lines $2x + 3y = -6$ and $x - 4y = 8$ intersect.

4. Find the equation of the line that passes through the points $(-2,4)$ and $(3,8)$.

5. Find the equation of a circle with center at $(-2,4)$ and passes through the point $(3,8)$.

6. Simplify $\sqrt{\dfrac{3}{8x^5y}}$.

7. Factor $81x^4 - 1$.

8. Combine $\dfrac{3a}{2b} - \dfrac{5a}{3c} + \dfrac{1}{a}$.

Chapter 10 summary

- **Dependent system** A system of n equations in n variables with more than one solution.
- **Inconsistent system** A system of n equations in n variables with no solution.
- **Recognizing dependent and inconsistent systems**
  - If we obtain a statement that is always true, such as $0 = 0$, the system of equations is dependent.
  - If we obtain a statement that is never true, such as $0 = 1$, the system of equations is inconsistent.
- **Row operations on matrices**
  1. If we multiply or divide each entry of a row by any nonzero value we do not change the solution set of the system.
  2. If we add a nonzero multiple of one row to a nonzero multiple of another row, and replace either row by the result, we do not change the solution set of the system of linear equations.
  3. Rearranging the order of the rows does not change the solution set of the system.
- **Matrix elimination** A method of solving systems of n equations in n variables. Each row serves as a key row once.
- **Identity matrix** The identity matrix of order n is the $n \times n$ matrix I such that $i_{a,b} = \begin{cases} 1 \text{ if } a = b \\ 0 \text{ if } a \neq b \end{cases}$, $1 \leq a, b \leq n$.
- **Minor** Given an nth order matrix A, the minor of element $a_{i,j}$ is the determinant of the $n - 1$ order matrix formed by deleting the ith row and jth column of matrix A.
- **Sign matrix of order n**

$$\begin{array}{cc} & n \text{ columns} \\ n \text{ rows} & \begin{bmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & \cdots & \\ + & - & + & - & \cdots & & \\ - & + & - & \cdots & & & \\ + & - & \cdots & & & & \\ - & \cdots & & & & & \\ \cdots & & & & & & \end{bmatrix} \end{array}$$

- **Determinant of an order 2 matrix**
$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$.
- **Determinant of an order n matrix, $n > 2$** Given an order n matrix, the determinant is the sum of the products of each element of any row or column, the corresponding element of the sign matrix, and its respective minor.
- The determinant of a system of n equations in n variables is 0 if and only if the system is dependent or inconsistent.
- **Cramer's rule** Given a system of n linear equations in the n variables v_i, $1 \leq i \leq n$, let D represent the determinant of the coefficient matrix, and let D_i be the determinant of the matrix composed of the coefficient matrix D with the ith column of D replaced by the column of constants. Then for each i, $1 \leq i \leq n$, $v_i = \dfrac{D_i}{D}$.
- **Linear programming problem in two variables** A problem that can be described by a nonempty set of linear inequalities, called constraints, and a linear equation in two variables, called the objective function. The constraints form a set of feasible solutions. The set of linear inequalities that correspond to the set of inequalities form a boundary to the feasible solutions.
- **Fundamental principle of linear programming** In a linear programming problem the objective function is always maximized or minimized at a vertex of the graph of feasible solutions.
- **Scalar** A real number.
- **Vector** A one-dimensional matrix.
- **Inverse of a matrix** For a square matrix C, if there exists a matrix C^{-1} such that $C^{-1}C = I$, and $CC^{-1} = I$, C^{-1} is the inverse of C and vice versa.
- A square matrix C has an inverse if and only if $|C| \neq 0$.

Chapter 10 review

[10–1] Solve the following systems of equations.

1. $3x + 2y = -6$
$-\frac{3}{2}x + y = 6$

2. $2x + \frac{3}{2}y = 5$
$-3x + \frac{1}{2}y = -2$

3. $-x + 4y = 20$
$2x + y = -22$

4. $0.4x + 0.5y = -2.9$
$-0.1x + y = 1.4$

5. $x - 5z = -7$
$-2x + y + 2z = 9$
$5x + 2y = -4$

6. $-4x - y + 3z = -7$
$-2x + 2y - z = -6$
$x + 2z = -3$

7. $x + y - z = 7$
$-2x + 4y + 2w = 4$
$-2y - 3z - 2w = 0$
$2x - 5z + 5w = 21$

8. $2x - z + 2w = -3$
$3x + z + 4w = 4$
$4y - 2z - w = -15$
$y + 2z + 2w = 6$

9. The perimeter of a certain rectangle is 182 centimeters; if the ratio of length to width is 8 to 5, find the values of the length and width.

10. The length of a certain rectangle is 2 inches less than twice the width. The perimeter is 158 inches. Find the two dimensions.

11. A total of \$15,000 was invested, part at 6% and the rest at 12%. The total income from both investments was \$1,530. How much was invested at each rate?

12. A parabola is the graph of an equation of the form $y = Ax^2 + Bx + C$. Find the values of A, B, and C so that the parabola will pass through the points $(-4,37)$, $(0,3)$, and $(2,10)$, and write the resulting equation.

[10–2] Solve the following systems of equations by matrix elimination; after describing the solution set, state dependent or inconsistent if appropriate.

13. $-2x + 5y = 7$
$2x + 3y = 9$

14. $2x + 5y = -17$
$-5x + \frac{2}{3}y = 3$

15. $-5x + y + 2z = 4$
$4x - y - z = -5$
$\frac{1}{2}x + 2y - 5z = 14$

16. $-x + 3y - 3z = 15$
$x + y - 3z = 7$
$-3x - y + 6z = -10$

17. $x - 3z + 5w = -9$
$-x + z + 3w = -7$
$3x - 3y + z = 15$
$-5x + y - 5z + 5w = -19$

18. $x + \frac{1}{2}y + 3z - 3w = -6$
$2x - \frac{3}{4}y + 3z + 5w = 16$
$-6x + y - w = -12$
$x - 2z = \frac{1}{3}$

19. For a certain electronics circuit Kirchhoff's law gives the system
$25i_1 + 20i_2 + 5i_3 = 50$
$40i_1 - 20i_2 - 10i_3 = 40$
$5i_1 + 10i_2 + 5i_3 = 45$
Find the currents i_1, i_2, and i_3.

20. A chemical company stores a certain herbicide in two concentrations of herbicide and water: 8% solution and 20% solution. It needs to manufacture 1,000 gallons of a 12% solution. How many gallons of each of the solutions should be mixed to obtain the required product?

[10–3] Solve the following systems of equations by Cramer's rule.

21. $-3x - 4y = 0$
$x + 9y = 4$

22. $5x + \frac{1}{3}y = 12$
$-8x + 2y = \frac{5}{8}$

23. $x + 8y + 3z = -4$
$x - 3y = 5$
$-x + 9y + 7z = -6$

24. $2x + 5y = -2$
$x - 5y - 2z = 1$
$4x + 2y = 0$

25. $2y + 3z - w = 2$
$-x + 2y + 5z = 0$
$-4x - 2z - 4w = -2$
$-4y - 4w = 1$

26. $x - 2y - 4w = 4$
$3x + y + 2z = -2$
$x - 2z + 6w = 0$
$4y - 2z + w = 2$

27. Solve for the variable D in the system:
$2A - B + 3C - D = 5$
$A + B - 2D + E = 0$
$-B - C = 10$
$3A - C + D + E = -4$
$C - E = -20$

28. If the three points that form the vertices of a triangle are (x_1,y_1), (x_2,y_2), (x_3,y_3), then it can be shown that the area of the triangle is the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Find the area of the triangle with vertices $(3\frac{1}{2},2)$, $(5,-3)$, and $(-2,8)$.

[10–4] Graph the solution set of the following linear inequalities.

29. $3x - 4y > 12$

30. $-12x + 6y < 18$

31. $x + 2y \geq -9$

32. $-9x + 3 > 4y$

33. $6y > 2$

34. $\frac{5}{3}y - \frac{7}{6} < 3x$

35. $2.4x - 1.2y \leq 4.8$

Graph the solution set of the following systems of linear inequalities.

36. $2x - y > 5$
$x + 2y < -6$

37. $3x + 2y \geq 12$
$2x + 2y < 9$

38. $\frac{1}{2}x + \frac{5}{4}y > 10$
$4x + 3y > -12$

In the following problems, maximize the value of the objective function P with regard to the constraints supplied. It is assumed that $x \geq 0$ and $y \geq 0$.

39. $-14x + 15y \leq 9$
$2x + 10y \leq 23$
$9x + 8y \leq 48$
$P = 4x + 2y$

40. $x + 6y \leq 18$
$x + 3y \leq 10$
$2x + y \leq 10$
$P = x + 3y$

41. A furniture factory is asked by its parent company to produce two products, tables and chairs. The factory's profit on tables is \$3 and on chairs is \$1. It takes 4 hours to assemble a table and $1\frac{1}{2}$ hours to assemble a chair. In a production run there are 300 hours available for assembly. It takes 2 hours to finish a table, and $\frac{5}{8}$ hour for a chair. The finishing department has 200 hours available in a production run. The factory may make any mix of tables and chairs it chooses since the parent company has other factories also. Maximize total profit from the production of tables and chairs for this factory.

42. A large logging company has crews of workers and supervisors. It has found that a crew of 1 supervisor and 4 loggers log 14 trees per day, and a crew of 3 supervisors and 18 loggers log 45 trees per day. The company has 40 supervisors and 200 loggers on its payroll. What mix of crews would produce the most trees per day? (For simplicity, assume that fractional parts of crews make sense.)

[10–5] Form the dot product of the given vectors.

43. $[3, -\frac{1}{4}][2, 5]$

44. $[-1, 10, -3, 2], [4, -2, -2, \frac{1}{2}]$

45. $[\frac{1}{4}, 1, -2], \begin{bmatrix} 4 \\ -4 \\ 5 \end{bmatrix}$

46. $\left[\sqrt{2}, \frac{1}{3}, -\frac{15}{\sqrt{\pi}}\right], \left[\sqrt{8}, 6, \frac{\pi}{5}\right]$

47. Find a vector v such that $[-2, 1, 6, 5] \cdot v = 3$.

Compute the indicated matrix products.

48. $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4 & -8 & 10 \\ 1 & 2 & -4 \end{bmatrix}$

49. $\begin{bmatrix} -x & 2 \\ 4y & -3 \end{bmatrix}\begin{bmatrix} 4x & 3 \\ y & 9 \end{bmatrix}$

50. $\begin{bmatrix} 2 & -3 & 1 \\ 4 & \frac{1}{2} & 6 \\ 0 & -2 & 3 \\ 10 & -1 & 2 \end{bmatrix}\begin{bmatrix} -4 & -8 & 10 & 1 \\ 2 & -4 & 3 & \frac{3}{4} \\ -1 & 0 & 4 & 1 \end{bmatrix}$

Find the inverse of each matrix. If the matrix does not have an inverse, state this.

51. $\begin{bmatrix} 2 & -5 \\ -3 & -1 \end{bmatrix}$

52. $\begin{bmatrix} 1 & -3 & 0 \\ 2 & 0 & -1 \\ 3 & 5 & 4 \end{bmatrix}$

Solve the system of equations using matrix multiplication.

53. $2x - 3y = 1$
$x + y = -2$

54. $x - y = 0$
$y + 3z = 2$
$-2x + y = -5$

Chapter 10 test

Solve the following systems of n equations in n variables by elimination.

1. $2x - 3y = -1$
$4x + 9y = 8$

2. $2x + 2y - z = 6$
$3x - 4y + z = 3$
$x - 2y + 3z = -2$

3. The length of a certain rectangle is 8 inches longer than three times the width. The perimeter is 208 inches. Find the two dimensions.

4. A total of \$12,000 was invested, part at 6% and the rest at 10%. The total income from both investments was \$1,020. How much was invested at each rate?

5. A parabola is the graph of an equation of the form $y = ax^2 + bx + c$. Find the equation of the parabola that will pass through the points $(-2,13)$, $(1,4)$, and $(2,9)$.

Solve the following systems of equations by matrix elimination.

6. $2x - y + 2z = 12$
$4x - y - z = 7$
$x - 2y - 5z = -9$

7. $2x - y + 2z = 4$
$x + 2y - w = -3$
$3x - 2y + z + w = 4$
$y - 5w = 3$

8. A company has two antifreeze mixtures on hand. One is a 30% solution (30% is alcohol) and the other is a 70% solution. How much of each should be mixed to obtain 500 gallons of a 45% solution?

Solve the following systems of equations by Cramer's rule.

9. $x + \frac{1}{3}y = 4$
$-x + 2y = \frac{5}{4}$

10. $2x + 5y + z = -2$
$x - 5y - 2z = 1$
$6x - 2z = -2$

11. If the three points that form the vertices of a triangle are (x_1,y_1), (x_2,y_2), (x_3,y_3), then it can be shown that the area of the triangle is the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
Find the area of the triangle with vertices $(3,-1)$, $(6,-3)$, and $(-2,8)$.

Graph the solution set of the linear inequalities.

12. $x + 2y \geq -10$

13. $5y - 40 < 10x$

14. Graph the solution to the system of linear inequalities.
$3x + y > 6$
$x - 2y \leq -6$

15. In the following problem, maximize the value of the objective function P with regard to the constraints supplied. Also assume $x \geq 0$ and $y \geq 0$.

$$2x + 6y \leq 18$$
$$x + 3y \leq 10$$
$$2x + y \leq 10$$
$$P = x + 2y$$

16. A company makes two mixtures of dog food nutrient supplement which it calls Regular and Prime. There are three ingredients, A, B, and C, in each mix. The table shows how much of each ingredient (in milligrams, mg) is in each gram of the food and the cost in cents for that mix. It also shows the minimum daily requirement (MDR) for each ingredient, in milligrams.

Mix	A	B	C	Cost per gram
Prime	2	5	8	6
Regular	2	3	1	3
MDR	6	12	8	

A dog owner wants to feed sufficient quantities of each supplement so that the owner's dog gets the MDR for each ingredient at the least expense. How many grams of each supplement should the owner feed the dog per day?

17. A large logging company has crews of workers and supervisors. It has found that a crew of 2 supervisors and 8 loggers log 22 trees per day, and a crew of 3 supervisors and 9 loggers log 31 trees per day. The company has 40 supervisors and 144 loggers on its payroll. What mix of crews would produce the most trees per day? (Assume that fractional parts of crews make sense.)

Form the dot product of the given vectors.

18. $[-1, 10, -3, 2], [4, -2, -2, \frac{1}{2}]$

19. $[\frac{1}{4}, 1, -3], \begin{bmatrix} -8 \\ 4 \\ -5 \end{bmatrix}$

Compute the indicated matrix products.

20. $\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & -4 \end{bmatrix}$

21. $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 6 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & 1 \\ 2 & -4 \end{bmatrix}$

Find the inverse of each matrix. If the matrix does not have an inverse, state this.

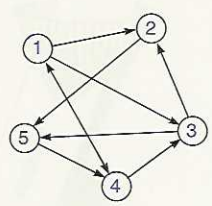
22. $\begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix}$

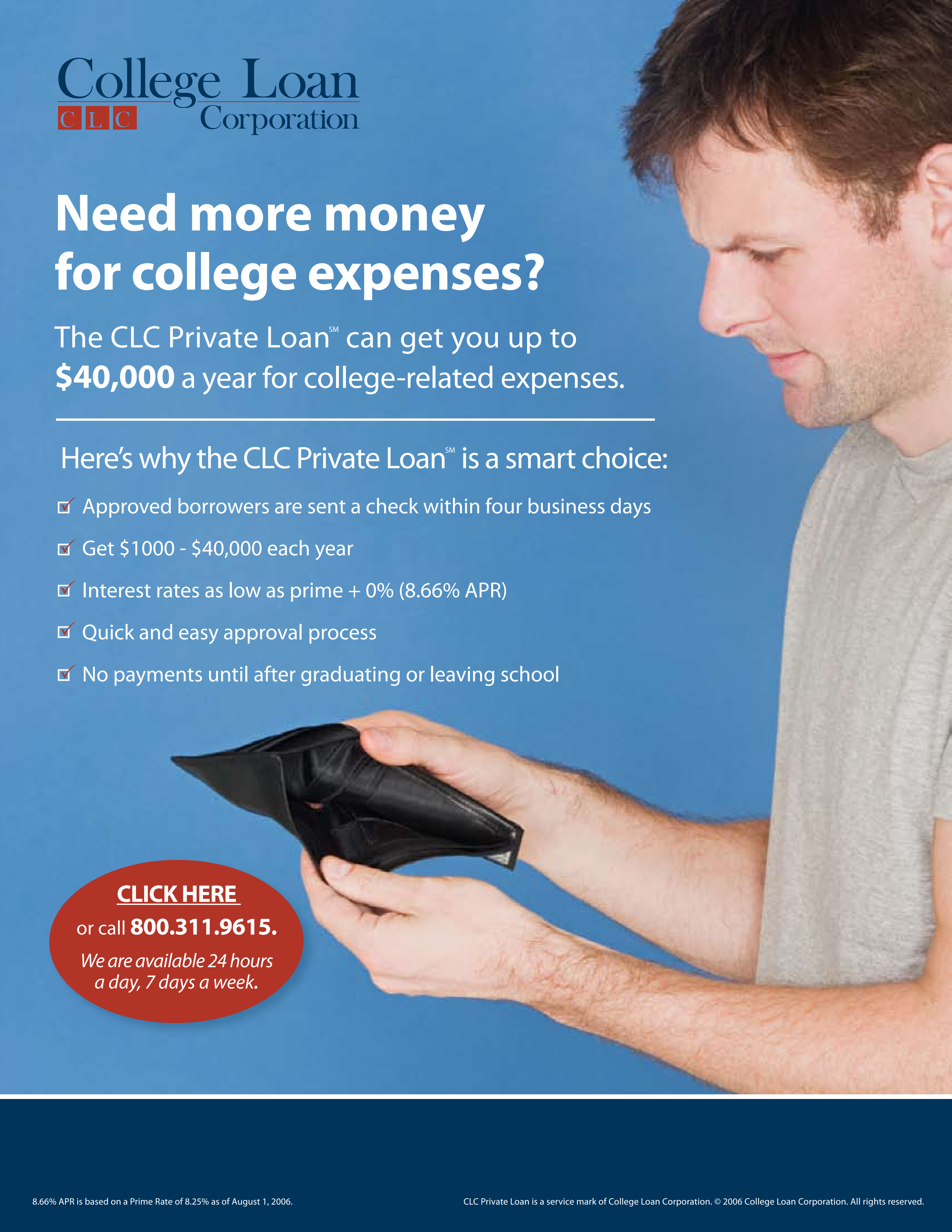
23. $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{bmatrix}$

24. Solve using matrix multiplication.
$$3x - 3y = 2$$
$$2x + y = -2$$

25. If $\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 5 & 18 \end{bmatrix}$, find a, b, c, and d.

26. The figure shows five points and the paths that connect them. Construct a 5×5 array A that shows the paths, then compute A^2 and use it to determine the number of paths of length 2 from node 1 to node 5.

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